Lecture 4

Thursday, January 29, 2015 1:40 PM

situation	$\forall x. P(x)$	$\exists x. \neg P(x)$	$\exists x. P(x)$	$\forall x. \neg P(x)$
no x of specified type	T	F	F	T
there are x of specified type & P is true for all such x	T	F	T	F
there are x of specified type & P is false for all such x	F	T	F	T
there are x of specified type & P is true for some of these, false for others	F	T	T	F
	$\underbrace{\hspace{4em}}_{\text{opposites}}$		$\underbrace{\hspace{4em}}_{\text{opposites}}$	

Thus we have proved

Prop The negation of $\forall x. P(x)$ is $\exists x. \neg P(x)$;
the negation of $\exists x. P(x)$ is $\forall x. \neg P(x)$ $\square$

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What do we prove? And how do we prove it?

$P \vee Q$: Suppose $P = F$, show then $Q = T$. (Or vice versa)

e.g. A pos prime # is odd or 2.

Pf Suppose p is a pos prime which is not odd. Then $p = 2 \cdot r$. Since p is prime, $r = 1$, so $p = 2$. $\square$

$P \Rightarrow Q$: Assume $P = T$, show then $Q = T$.

Or by "contrapositive": Assume $Q = F$, show then $P = F$.

e.g. If an integer n is a mult of 2 & mult of 3, then n is a mult of 6.

Pf Let n be a mult of 2 & 3. Then $n = 2p = 3q$ for some integers p, q. $2p = 3q$ forces q even so that $q = 2r$. for int r
Thus $n = 3 \cdot 2 \cdot r = 6r$, and we see n is a mult of 6. $\square$

$\forall x. P(x)$: Let x be arbitrary of specified type. Prove $P(x) = T$.

e.g. All real #s x satisfy $x^2 = (-x)^2$.

Pf Let x be an arb real #. Then $(-x)^2 = (-x)(-x) = (-1)x(-1)x$
$= (-1)(-1)xx = 1 \cdot x^2 = x^2$, as desired. $\square$

$\exists x. P(x)$: Find or construct x such that $P(x) = T$.

Or invoke a theorem guaranteeing the existence of such x.

e.g. There exists real # x s.t. $x^3 - 3x = 2$.

Pf. Let $x = 2$ and observe that $2^3 - 3 \cdot 2 = 8 - 6 = 2$, as desired. $\square$

e.g. There exists real x s.t. $x^3 - x = 1$.

Pf $f(x) = x^3 - x$ is cts & $f(0) = 0$, $f(2) = 6$. Since $0 < 1 < 6$, IVT implies $\exists x.\ 0 < x < 2$ and $f(x) = 1$.

Thm [Euclid, c300 BCE] There are infinitely many prime #s.

Pf (by contradiction) Suppose there are only finitely many prime #s $p_1, p_2, \ldots, p_n$. Let $a = p_1 p_2 \cdots p_n + 1$. Since 2 is a prime, $a > 1$. By FTA, a has a prime factor $p = p_i$ for some $1 \le i \le n$. Since p divides a & p divides $p_1 p_2 \cdots p_n$, it must divide $a - p_1 p_2 \cdots p_n = 1$, which is absurd. Thus there are in fact ∞ly many primes. $\square$

In order to effect this, we had to negate a statement:

statement	negation
P	$\neg P$
$P \wedge Q$	$\neg(P \wedge Q) = \neg P \vee \neg Q$
$P \vee Q$	$\neg(P \vee Q) = \neg P \wedge \neg Q$
$P \Rightarrow Q$	$\neg(P \Rightarrow Q) = P \wedge (\neg Q)$
$\forall x.\ P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	$\forall x.\ \neg P(x)$.