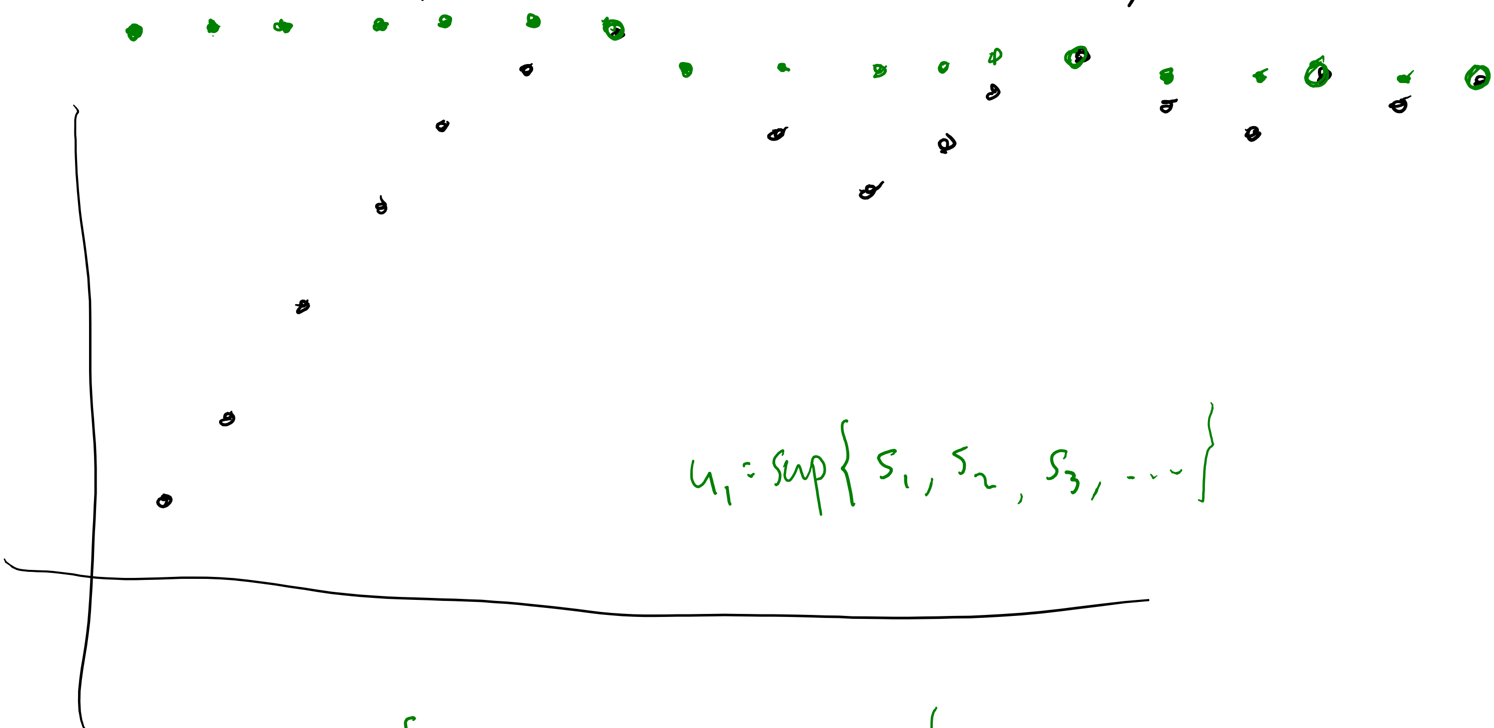
Lecture 39

Friday, April 10, 2015 8:02 AM

(s_n) is a Cauchy sequence $\forall \varepsilon > 0 \ \exists N > 0$ s.t. if $n, m > N$, then $|s_m - s_n| < \varepsilon$.

Thm If (s_n) is a Cauchy seq in $\mathbb{R}$ (or $\mathbb{C}$) then $(s_n) \longrightarrow L \in \mathbb{R}$ (or $\mathbb{C}$)



$u_n = \sup\{s_n, s_{n+1}, s_{n+2}, \ldots\}$

① $(u_n) \longrightarrow L = \inf\{u_i \mid i \geq 1\}$ b/c u_n is a bdd below non-increasing seq.

② $\lim s_n = L$

<u>Subsequences</u> A subsequence of a sequence (s_n) is $(s_{k_1}, s_{k_2}, s_{k_3}, \ldots)$ where $k_i \in \mathbb{Z}^+$ &

$$1 \le k_1 < k_2 < k_3 < \cdots$$

In other words $k : \mathbb{Z}^+ \to \mathbb{Z}^+$ is strictly increasing, and $(s_{k_i}) = s \circ k$.

<u>E.g.</u> $(s_2, s_4, s_6, s_8, \ldots)$ is a subsequence of $(s_1, s_2, s_3, \ldots)$. Here $k_i = ?$

$k_1 = 2$, $k_2 = 4$, $k_3 = 6$, ...

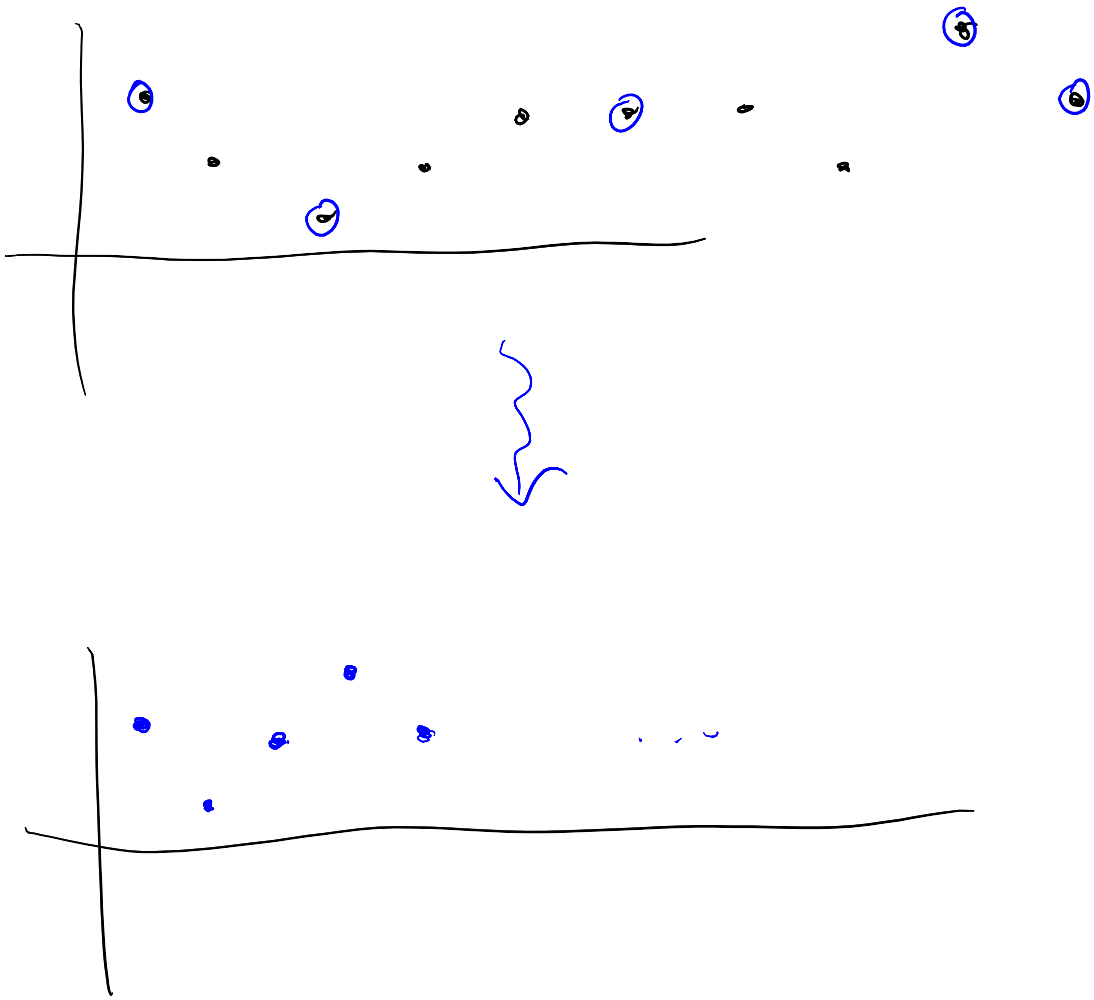
$k_i = 2i$.

<u>E.g.</u> $(s_n) = \left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$

subsequences include $\left(\frac{1}{2n}\right) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right)$

$\left(\frac{1}{3n}\right) = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \ldots\right)$

$\left(\frac{1}{n^2}\right) = \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right)$



Thm ① A subsequence of a convergent sequence is convergent w/ the same limit.

② A subsequence of a Cauchy sequence is Cauchy.

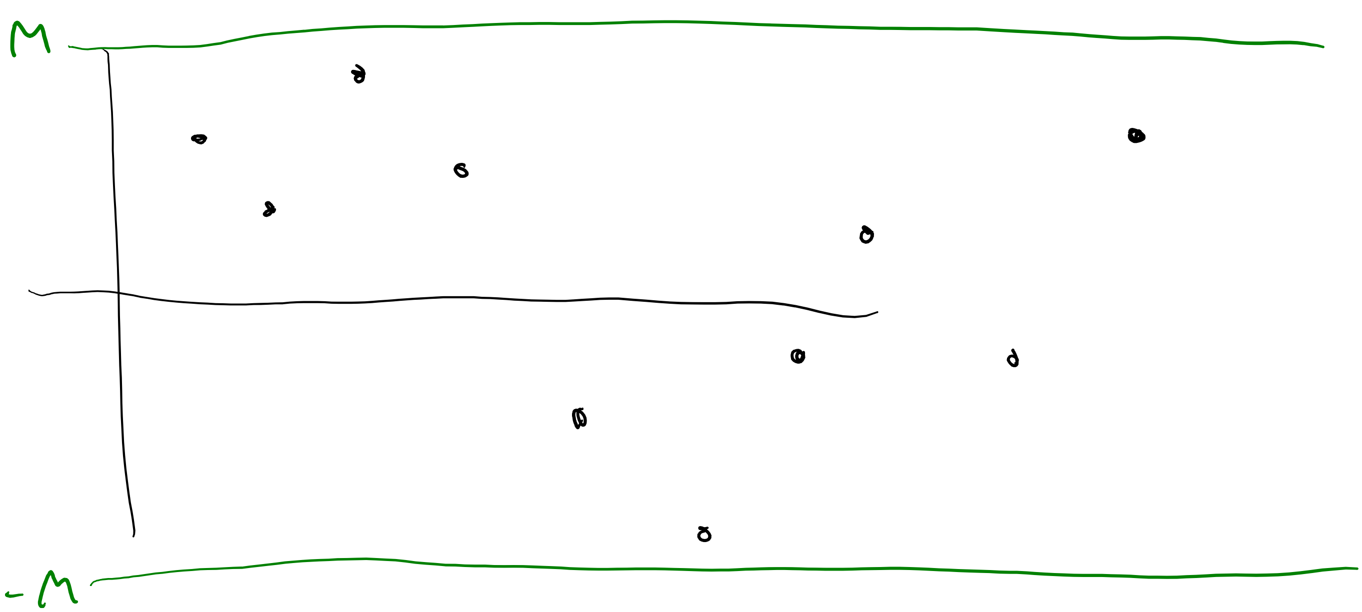
Pf ② Let (s_n) be a Cauchy seq & (s_{k_i}) a subsequence of (s_n). Let $\varepsilon > 0$. Since (s_n) is Cauchy, $\exists N > 0$ s.t. if $n, m > N$, then $|s_m - s_n| < \varepsilon$.

Note that $k_n \geq n$ & $k_m \geq m$. Thus, for any $n, m > N$, $k_n, k_m > N$. Thus

$$\left| s_{k_{\cancel{n}\,m}} - s_{k_{\cancel{m}\,n}} \right| < \varepsilon$$ by the above Cauchy condition for (s_n). $\square$

<u>Thm</u> Every bounded sequence has a Cauchy subsequence.

<u>Pf</u> (s_n) bdd seq of real #s. $M > 0$ s.t. $|s_n| \leq M$ $\forall n$. Set $a_0 = -M$, $b_0 = M$ and note $s_n \in [a_0, b_0]$ $\forall n$. Set $l_0 = 0$.



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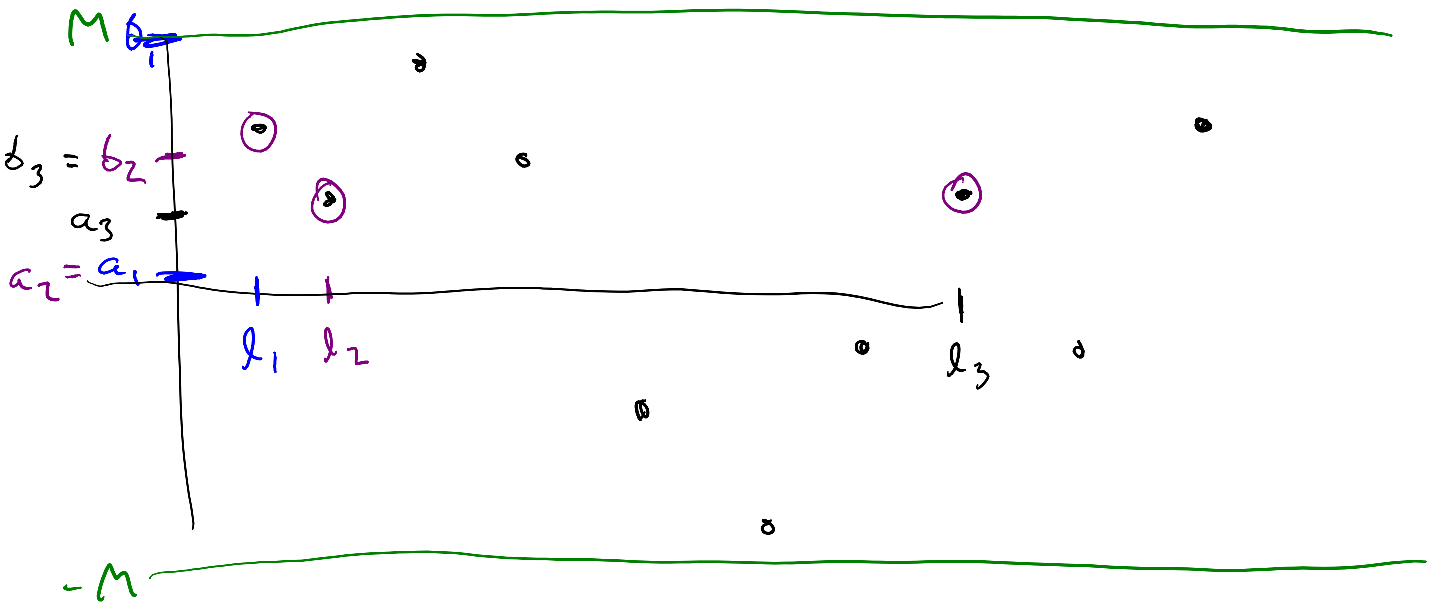

Goal: Construct $[a_0, b_0] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$ w/ $b_m - a_m = 2^{-m} \cdot 2M$ and each $[a_m, b_m]$ containing infinitely many elts of the sequence.

To achieve goal: choose either the first or second half of $[a_{m-1}, b_{m-1}]$ — whichever contains ∞ly many terms.

Let l_m be the first integer $> l_{m-1}$ s.t. $s_{l_m} \in [a_m, b_m]$

Then $(s_{\ell_n})_n$ is a subseq of (s_n).

Let $\varepsilon > 0$. $\exists N > 0$ s.t. $\frac{1}{2^N} < \frac{\varepsilon}{2M}$. Then if $m, n > N$,

$s_{\ell_m}, s_{\ell_n} \in [a_N, b_N] \Rightarrow |s_{\ell_m} - s_{\ell_n}| \le b_N - a_N = 2^{-N} 2M < \varepsilon$.

$\mathbb{C}$-seq: Re & Im parts... $\square$

<u>Cor</u> Every bdd seq in $\mathbb{C}$ has a convergent subseq.

<u>Defn</u> A <u>subsequential limit</u> of a seq (s_n) is a limit of any subseq of (s_n).

(s_n)	{subseq limits of (s_n)}
conv	$\{\lim s_n\}$
$((-1)^n)$	$\{1, -1\}$
$((-1)^n + \frac{1}{n})$	$\{1, -1\}$
$((-1)^n n + n + 3)$	$\{3, \infty\}$
(i^n)	$\{1, i, -i, -1\}$
(n)	$\{\infty\}$
$((-1)^n n)$	$\{\infty, -\infty\}$