Lecture 37

Monday, April 6, 2015 11:49 AM

Cauchy sequences

Defn A sequence (s_n) is Cauchy if $\forall \varepsilon > 0$ $\exists N > 0$ s.t. if $m, n > N$, then

$$|s_m - s_n| < \varepsilon.$$

Interpretation: terms of s_n get close to each other, but no limit referenced.

Thm Every Cauchy sequence is bounded.

Pf Let (s_n) be a Cauchy seq. Take $\varepsilon = 1 > 0$. Then $\exists N > 0$ s.t. if $m, n > N$, $|s_m - s_n| < 1$. Then $\{|s_1|, |s_2|, \dots, |s_{N+1}|\}$ is a finite set in $\mathbb{R}$ and hence a bdd subset, say M' an upper bound. Set $M = M' + 1$. Then $|s_1|, \dots, |s_N| < M$ and for $n > N$, $|s_n| = |s_n - s_{N+1} + s_{N+1}| \leq |s_n - s_{N+1}| + |s_{N+1}|$ $< 1 + M' = M$ b/c $n, N+1 > N$. Thus $\{|s_i|\}_{i \geq 1}$ is bdd $\Rightarrow$ (s_n) is bdd. $\square$

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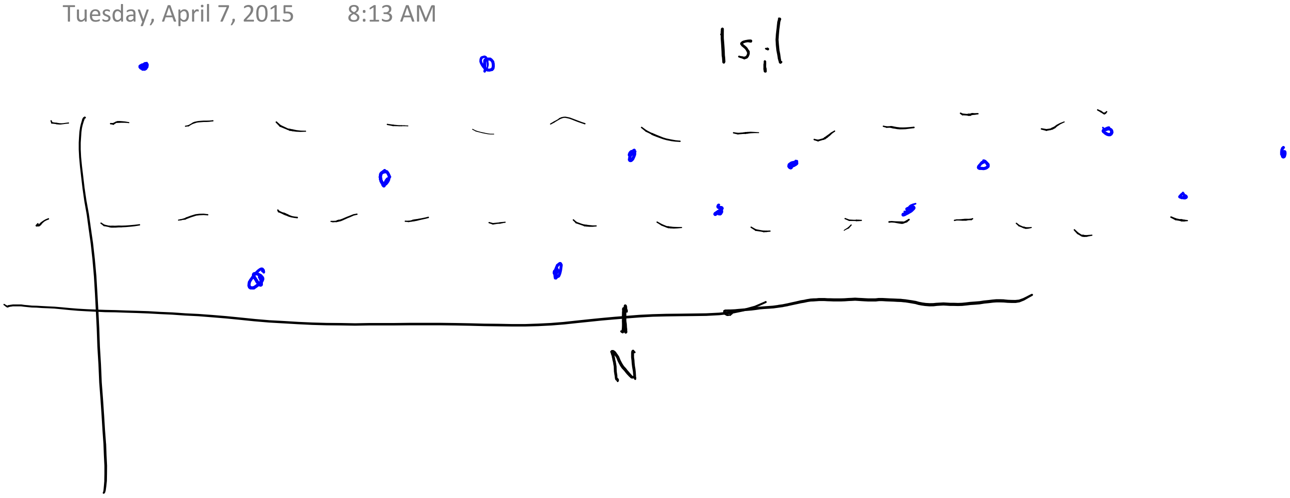
Tuesday, April 7, 2015 8:13 AM

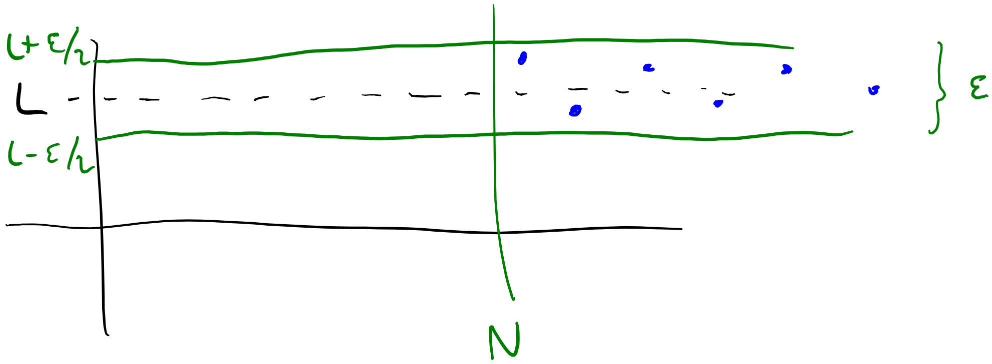


<u>Thm</u> Every convergent sequence is Cauchy.

Pf Suppose $(s_n) \to L$. Let $\varepsilon > 0$. Then $\exists N > 0$ s.t. if $n > N$, $|s_n - L| < \frac{\varepsilon}{2}$. Thus for $m, n > N$,

$$
\begin{aligned}
|s_m - s_n| &= |s_m - L + L - s_n| \\
&\leq |s_m - L| + |s_n - L| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .
\end{aligned}
$$

Thus (s_n) is Cauchy. $\square$



<u>Thm</u> Every Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$ is convergent.

<u>Note</u> Sequences of rational #s which are Cauchy need not converge to an elt of $\mathbb{Q}$.

<u>Pf</u> First let (s_n) be a Cauchy seq of real #s. Then $\{s_n, s_{n+1}, s_{n+2}, \dots\}$ is bounded so by completeness of $\mathbb{R}$, $u_n = \sup\{s_n, s_{n+1}, \dots\}$ exists in $\mathbb{R}$. Note that $\forall n$, $u_n \geq u_{n+1}$.

(If $S \subseteq T \subseteq \mathbb{R}$ have suprema, then $\sup(T) \geq \sup(S)$ & $\{s_n, s_{n+1}, \dots\} \supseteq \{s_{n+1}, s_{n+2}, \dots\}$)

Then $\{u_1, u_2, u_3, \dots\}$ is bdd below b/c $\{s_1, s_2, \dots\}$ is bdd below, thus (u_n) is bdd below and non-increasing sequence, so

$$\lim_{n\to\infty} u_n = \inf\{u_1, u_2, \dots\} =: L$$

Claim $L = \lim_{n\to\infty} s_n$.

Let $\varepsilon > 0$. $\exists N_1 > 0$ s.t. for $m \geq n > N_1$,

$|s_n - s_m| < \varepsilon/2$. Fix $n > N_1$. For $m \geq n$,

$$s_m < s_n + \varepsilon/2 \implies s_m \leq u_n < s_n + \varepsilon/2.$$

This holds for $m = n$, so $|s_n - u_n| < \varepsilon/2$.

Since $L = \inf\{u_i\}_{i \geq 1}$, $\exists N_2 > 0$ s.t. $0 \leq u_{N_2} - L < \varepsilon/2$

Set $N = \max\{N_1, N_2\}$. Let $n > N$. Then

$$L \leq u_n \leq u_N \leq u_{N_2} \quad \text{so} \quad 0 \leq u_n - L \leq u_{N_2} - L < \varepsilon/2.$$

Hence
$$\begin{aligned} |s_n - L| &= |s_n - u_n + u_n - L| \\ &\leq |s_n - u_n| + |u_n - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

thus $\lim_{n\to\infty} s_n = L$.

If (s_n) is a Cauchy seq of cpx #s, then

(Claim) $(\mathrm{Re}(s_n))$ & $(\mathrm{Im}(s_n))$ are Cauchy.

Moreover $(s_n) \longrightarrow \left(\lim_{n\to\infty} \mathrm{Re}(s_n)\right) + \left(\lim_{n\to\infty} \mathrm{Im}(s_n)\right) i$.

$\blacksquare$

Dedekind complete — old notion of completeness

Cauchy complete — every Cauchy sequence has a limit.

e.g. Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Claim (s_n) is not Cauchy, does not converge, is monotone, and is not bdd:

If $m > n$, $s_m - s_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m}\right)$

$$- \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$= \underbrace{\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{n+1}}_{m-n \text{ terms each } \geq \frac{1}{m}}$$

$$\geq \frac{m-n}{m} = 1 - \frac{n}{m}$$

If, e.g., $m = 3n$, then $1 - \frac{n}{m} = 1 - \frac{n}{3n} = \frac{2}{3} > \frac{1}{2}$.

So if we take $\varepsilon = \frac{1}{2}$, we'll see it's impossible to satisfy the Cauchy condition!

So (s_n) is not Cauchy $\Longrightarrow$ (s_n) does not converge.

But (s_n) is strictly increasing. Thus, if (s_n) is bdd above, it converges, . So (s_n) is unbdd.