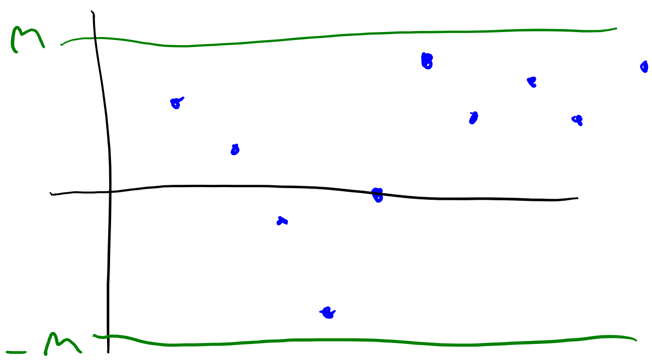
Lecture 36

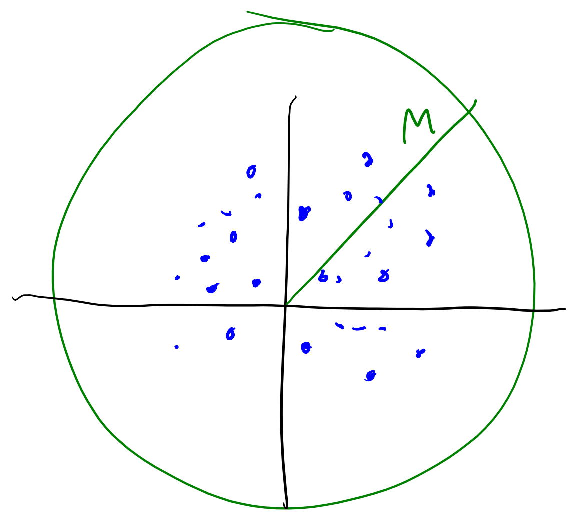
Monday, April 6, 2015 8:01 AM

The ratio test for sequences

Convenient criterion for $s_n \longrightarrow 0$.

Defn A sequence is *bounded* if $\exists M > 0$ s.t. $|s_n| < M$ for every $n \in \mathbb{Z}^+$.





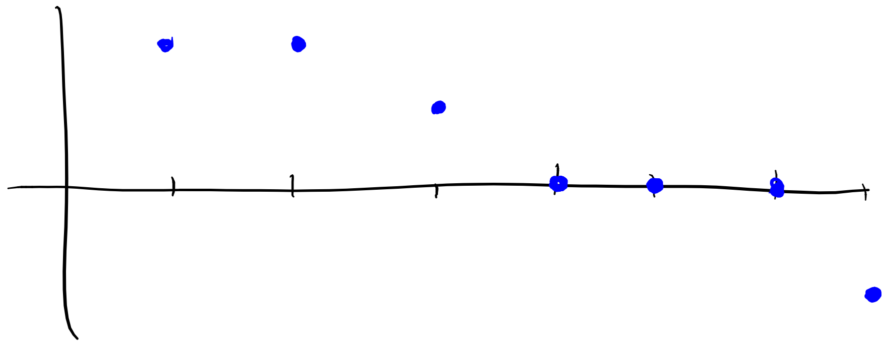
e.g.
- $(\sin(n))_{n \geq 1}$
- $\left(\frac{1}{n}\right)_{n \geq 1}$
- $((-1)^n)_{n \geq 1}$
- $\left(\frac{\zeta^n}{n}\right)_{n \geq 1}$, $\zeta \in \mathbb{C}$ w/ $|\zeta| = 1$, $\Theta(\zeta) = \frac{2\pi}{3}$

Defn A sequence of real ~~numb~~ numbers (s_n) is

$$\left\{\begin{array}{l} \text{non-decreasing} \\ \text{non-increasing} \\ \text{strictly increasing} \\ \text{strictly decreasing} \end{array}\right. \quad \text{if for all } n \in \mathbb{Z}^+ \quad \left\{\begin{array}{l} s_n \leq s_{n+1} \\ s_n \geq s_{n+1} \\ s_n < s_{n+1} \\ s_n > s_{n+1} \end{array}\right.$$

Monday, April 6, 2015 8:12 AM

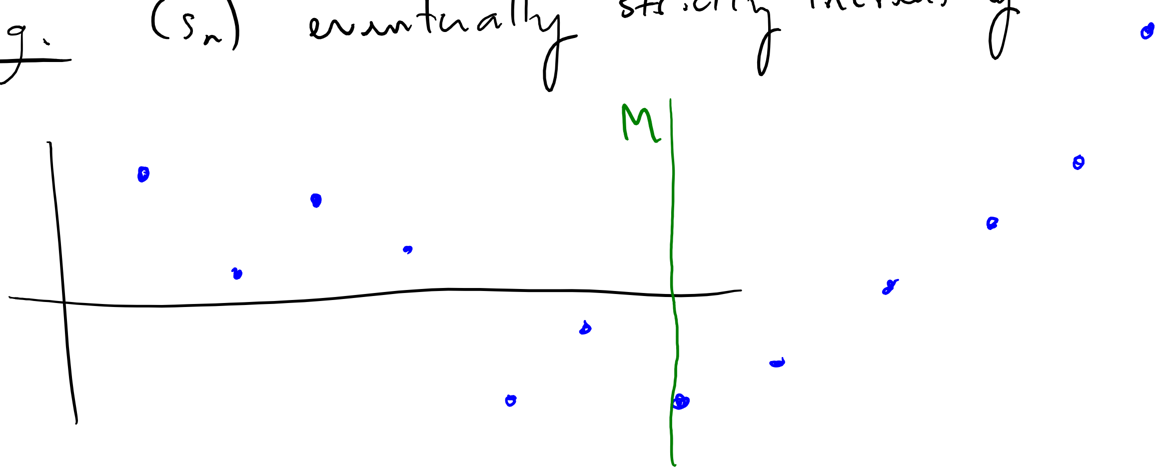
e.g. (s_n) non-increasing looks like



(s_n) is monotone if one of the above conditions obtains.

If $\mathbb{X}$ is one of the above conditions, we say a sequence is eventually $\mathbb{X}$ if $\exists M>0$ s.t. $(s_n)_{n\geq M}$ is $\mathbb{X}$.

e.g. (s_n) eventually strictly increasing:



Thm Let (s_n) be a bounded sequence of real numbers which is eventually non-decreasing (respectively non-increasing). Then $\lim s_n$ exists and equals $\sup\{s_N, s_{N+1}, s_{N+2}, \ldots\}$ (resp. $\inf\{s_N, s_{N+1}, s_{N+2}, \ldots\}$) where $(s_n)_{n \geq N}$ is non-decreasing (resp. non-increasing).

Pf Suppose s_n is non-decreasing for $n \geq N$ and (s_n) is bounded. Then $\{s_N, s_{N+1}, \ldots\}$ is bounded above so by completeness of $\mathbb{R}$ has a supremum in $\mathbb{R}$; call it L.

Let $\varepsilon > 0$. Then $\exists$ positive integer $N' > N$ s.t.

$\underbrace{0 \leq L - s_{N'} < \varepsilon}_{s_{N'} \text{ gets } \varepsilon \text{ close to } L}$. Hence $\forall\, n > N'$, $s_{N'} \leq s_n$ so

$0 \leq L - s_n \leq L - s_{N'} < \varepsilon$. Thus $|s_n - L| < \varepsilon$

for $n > N'$ so $\lim\limits_{n \to \infty} s_n = L$. $\square$

Thm [Ratio test for sequences] Let (s_n) be a sequence of complex numbers s.t. $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$ and $L<1$. Then $\lim_{n\to\infty} s_n = 0$.

Cor If $r \in \mathbb{C}$ has $|r|<1$, then $\lim_{n\to\infty} r^n = 0$.

e.g. $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

$\lim_{n\to\infty} \left(\frac{\pi}{3}\right)^n = 0$ b/c $\left|\frac{\pi}{3}\right| = \frac{1}{3} < 1$.

"

$\lim_{n\to\infty} \frac{\pi^n}{3^n}$

Pf of Cor $\left|\frac{r^{n+1}}{r^n}\right| = |r| < 1$ so ratio test implies $\lim r^n = 0$. $\square$

Pf of ratio test Let $\varepsilon>0$ and choose r strictly between L and 1. Note that r & $r-L$ are positive numbers. Since $\lim \left|\frac{s_{n+1}}{s_n}\right| = L$, $\exists N_1>0$ s.t. $\forall n>N_1$,

$$\left|\left|\frac{s_{n+1}}{s_n}\right| - L\right| < r-L.$$

Thus $$\left|\frac{s_{n+1}}{s_n}\right| = \left|\left|\frac{s_{n+1}}{s_n}\right| - L + L\right|$$

$$\leq \left|\left|\frac{s_{n+1}}{s_n}\right| - L\right| + |L| \qquad (\Delta \text{ ineq})$$

$$< r - L + L \qquad \text{for } n > N_1$$

$$= r$$

<u>Claim</u> Let n_0 be the smallest positive integer $\geq N_1$. Then for every $n \geq n_0$,

$$|s_n| \leq r^{n-n_0}|s_{n_0}|$$

Pf of claim by induction on n:

$$\left|\frac{s_{n_0+1}}{s_{n_0}}\right| < r \implies |s_{n_0+1}| < r|s_{n_0}| = r^{n_0+1-n_0}|s_{n_0}|$$

$$|s_{n_0}| = r^0|s_{n_0}| = r^{n_0-n_0}|s_{n_0}| \qquad \text{base case ✓}$$

For some $n > n_0$ assume $|s_{n-1}| \leq r^{n-1-n_0}|s_{n_0}|$.
(for induction)

Then $\left|\frac{s_n}{s_{n-1}}\right| < r$ by above work.

so $|s_n| < r\,|s_{n-1}| \le r\, r^{n-1-n_0}|s_{n_0}|$ by ind hyp

$= r^{n-n_0}|s_{n_0}|$.

Thus the claim holds by induction.

Now note $0 < r < 1 \Rightarrow r^{n+1} < r^n \Rightarrow (r^{n-n_0})_{n \ge n_0}$ is strictly decreasing & bdd below by 0. & above by 1.

Let $K = \inf\{r^{n-n_0} \mid n \ge n_0\}$ which exists and is nonnegative. By today's first thm,

$K = \lim r^{n-n_0}$.

Suppose for ⨳ that $K > 0$. Consider $\frac{K(1-r)}{r}$.

$\exists N > n_0$ s.t. $0 \le r^n - K < \frac{K(1-r)}{r}$

$\Rightarrow 0 \le r^{n+1} - Kr < K(1-r)$

$0 \le r^{n+1} < K$.

Monday, April 6, 2015 8:53 AM

But $K \le r^{N+1} < K$, which is a .

Thus $K = 0 = \lim r^{n-n_0}$.

Now $\lim r^{n-n_0}|s_{n_0}| = \left(\lim r^{n-n_0}\right)\cdot\left(\lim |s_{n_0}|\right)$

$$= 0 \cdot |s_{n_0}|$$

$$= 0$$

and $0 \le |s_n| \le r^{n-n_0}|s_{n_0}| \longrightarrow 0$

so the squeeze theorem $\Rightarrow \lim s_n = 0$.

$\square$