Lecture 34

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Claim If $f: \mathbb{R} \longrightarrow \mathbb{Q}$ is cts, then f is constant.

Pf Consider the fn $\tilde{f} = i \circ f$ where $i: \mathbb{Q} \longrightarrow \mathbb{R}$ is the standard inclusion. Since i & f are cts, $\tilde{f}$ is cts. Suppose for ~~Q~~ $\tilde{f}$ is not constant. Then $\exists\, x, y \in \mathbb{R}$ s.t. $x \neq y$ and $\tilde{f}(x) \neq \tilde{f}(y)$. But then $\exists$ an irrational z between $\tilde{f}(x)$ & $\tilde{f}(y)$ and, by the intermediate value theorem, c between x & y s.t. $\tilde{f}(c) = z$. ~~But~~ $\tilde{f}(c) = f(c)$, and $f(c)$ must be rational. Thus $\tilde{f}$ (and hence f) is constant. $\square$

Divergence of sequences.

Def'n A sequence (s_n) **diverges** if $\forall L \in \mathbb{C}$, $\lim_{n\to\infty} s_n \neq L$.
We say that (s_n) is **divergent**

$$\forall L \in \mathbb{C},\ \neg\Big(\forall \varepsilon > 0\ \exists N > 0 \text{ s.t. if } n > N, \text{ then } |s_n - L| < \varepsilon\Big)$$

$$= \forall L \in \mathbb{C}\ \exists \varepsilon > 0 \text{ s.t. } \forall N > 0\ \exists n > N \text{ s.t. } |s_n - L| \geq \varepsilon.$$

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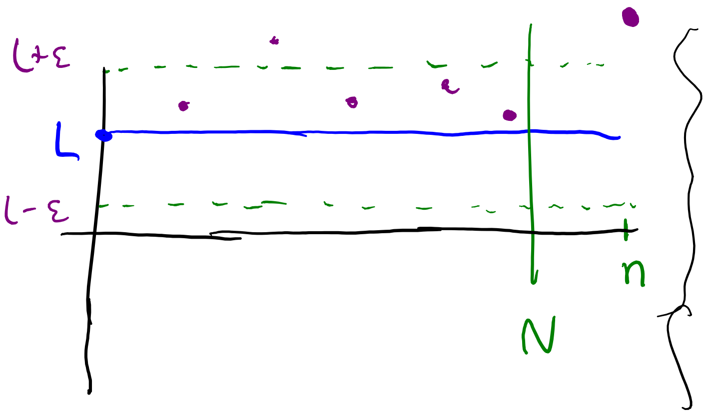
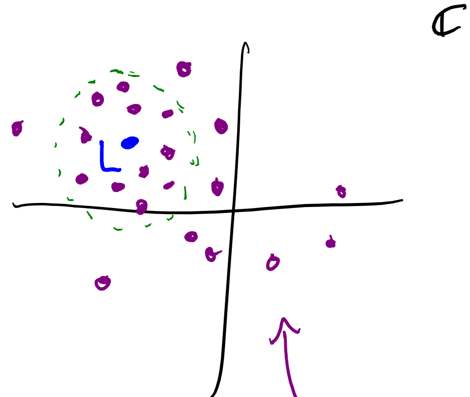
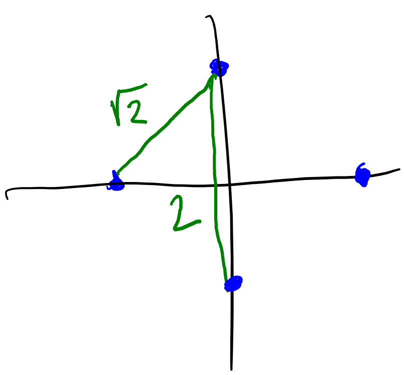
$$\forall L \in \mathbb{C} \ \ \exists \varepsilon > 0 \text{ s.t. } \forall N > 0 \ \ \exists n > N \text{ s.t.}$$
$$|s_n - L| \geq \varepsilon$$





∞ly many pts outside $B(L, \varepsilon)$.

e.g. $(s_n) = (i^n) = (i, -1, -i, 1, i, -1, -i, 1, \ldots)$



Claim (s_n) diverges.

Pf Given $L \in \mathbb{C}$ set $\varepsilon = 1$.
Note that at least 1 of $i, -1, -i, 1$ is outside of $B(L, 1)$. If $s_i \notin B(L, 1)$ then $s_j \notin B(L, 1)$ $\forall j \equiv i \pmod 4$. Since there are infinitely many such j, we always have $s_n \notin B(L, 1)$ for some

$j - i$ is a multiple of 4, i.e. $[i] = [j] \in \mathbb{Z}/4\mathbb{Z}$

$[i] = [j] \in \mathbb{Z}/4\mathbb{Z}$

we always have $s_n \notin B(L, 1)$ for some $n >$ fixed N.

e.g. ζ = cpx # w/ $|\zeta| = 1$ and $\Theta(\zeta) = \frac{2\pi}{3}$.

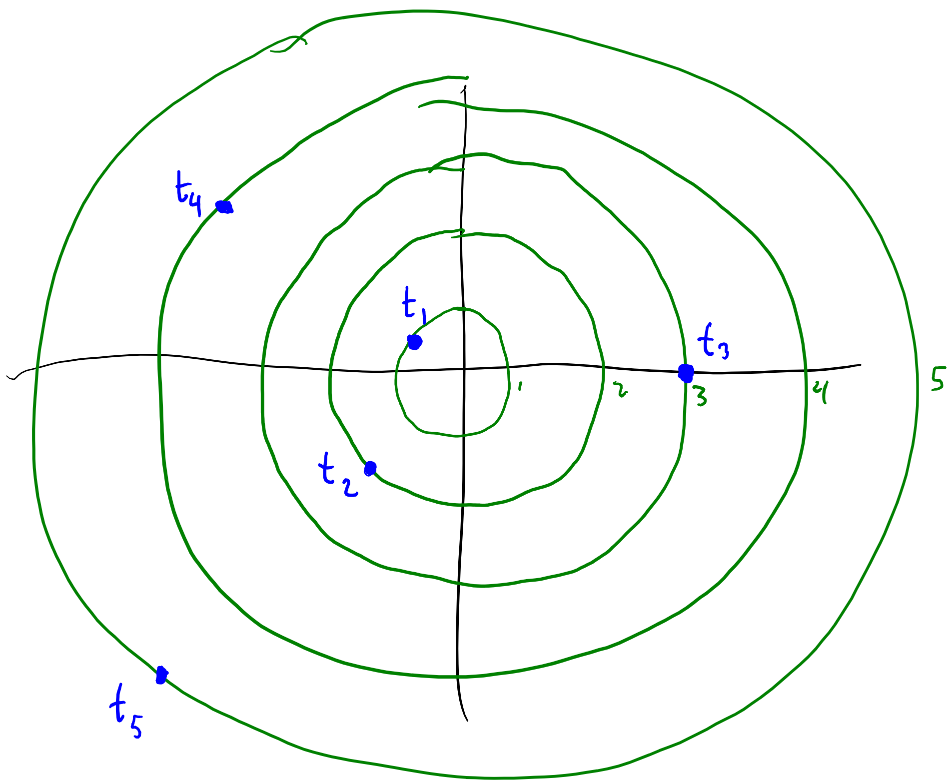
(ζ = zeta)

Note ζ^3 has $|\zeta^3| = |\zeta|^3 = 1$

& $\Theta(\zeta^3) = 3 \cdot \Theta(\zeta) = 2\pi$

$\Longrightarrow \zeta^3 = 1$.

Define $(t_n) = (n \cdot \zeta^n)$.



$t_4 = 4 \cdot \zeta^4$

$= 4 \cdot \zeta^3 \cdot \zeta$

$= 4\zeta$

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Claim $(t_n) = (n\zeta^n)$ diverges.

Pf Given $L \in \mathbb{C}$ set $\varepsilon = 42$. Then if $N > 0$, take $n > N$ s.t. $n - |L| > 42$. Then

$$|t_n - L| \geq |t_n| - |L| = |n\zeta^n| - |L|$$

$$= |n||\zeta|^n - |L|$$

$$= n - |L|$$

$$> 42. \quad \square$$