Lecture 32

Monday, March 30, 2015 8:02 AM

Sequences

Motivation If f has continuous $(n+1)$-st derivative, then

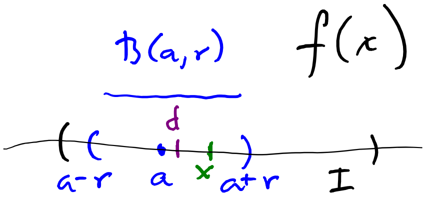
$$T_{n,f,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the Taylor polynomial of f centered at $x=a$.

As $n \to \infty$, $T_{n,f,a}(x)$ becomes a better & better approximation of f:

Taylor's Remainder Theorem $I \subseteq \mathbb{R}$ an interval, $r>0$ s.t. $B(a,r) \subseteq I$, $f: I \to \mathbb{R}$ with cts $f^{(k)}(x)$, $k=0,1,\dots,n+1$. Then $\forall x \in B(a,r)$ $\exists d$ between x and a s.t.

$$f(x) = T_{n,f,a}(x) + \frac{f^{(n+1)}(d)}{(n+1)!}\underbrace{(x-a)^{n+1}}_{\text{gets very small as } n\to\infty \text{ if } x \text{ is near } a}$$



We would hope that there is a way to recover f itself when $n \to \infty$:

$$f(x) \stackrel{?}{=} T_{\infty, f, a}(x) = \underbrace{\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{\text{What is this??}}$$

To understand infinite sums of fns we need to understand infinite sums of numbers for which we need to understand $\underbrace{\text{infinite lists of numbers}}_{\text{sequences!}}$.

__Defn__ A __sequence__ is a function $s : \mathbb{Z}^+ \to \mathbb{C}$.

Write s_n for $s(n)$ and call this the n-th term of s.

Write $s = (s_n) = (s_n)_{n \in \mathbb{Z}^+} = (s_n)_{n=1}^{\infty} = (s_1, s_2, s_3, \dots)$

$\{s_n\}$

e.g.

- $(0) = (0, 0, 0, \ldots)$
- For $c \in \mathbb{C}$, $(c) = (c, c, c, \ldots)$
- $\left((-1)^n\right)_{n=1}^{\infty} = \left((-1)^1, (-1)^2, (-1)^3, (-1)^4, \ldots\right)$

$$= (-1, 1, -1, 1, \ldots)$$

- $\left(i^n\right)_{n=1}^{\infty} = \left(i^1, i^2, i^3, i^4, \ldots\right)$

$$= (i, -1, -i, 1, i, -1, -i, 1, \ldots)$$

- $\left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right)$

Recursively defined sequences:

Define $F_1 = 1$, $F_2 = 1$, and $n > 2$ define $F_n = F_{n-2} + F_{n-1}$. Get the Fibonacci sequence:

$$(F_n) = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)$$

- $(n) = (1, 2, 3, 4, \dots)$

  ↳ $id : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ which is a bijection

Whenever $s : \mathbb{Z}^+ \longrightarrow A$ is a bijection we call A a <u>countable set</u>; s <u>enumerates</u> A.

<u>Q</u> Is $\mathbb{Z}$ countable?

$$((-1)^n \cdot n) = (-1, 2, -3, 4, -5, 6, \dots)$$

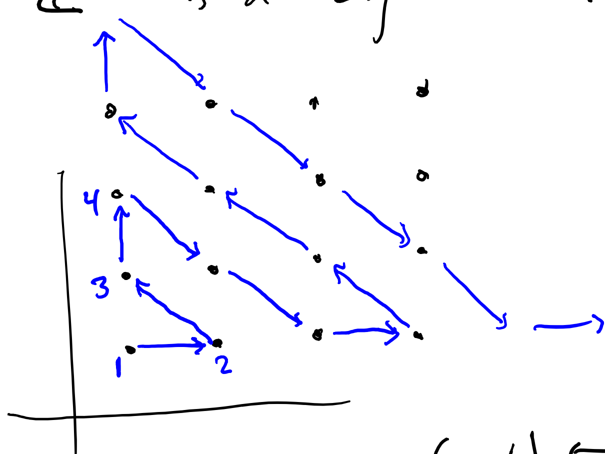
not quite...

$$(0, 1, -1, 2, -2, 3, -3, 4, -4, \dots)$$

$$: \mathbb{Z}^+ \longrightarrow \mathbb{Z} \quad \text{is a bijection!}$$

- $\mathbb{Z}^+ \times \mathbb{Z}^+$?

Following the blue arrows enumerates $\mathbb{Z}^+ \times \mathbb{Z}^+$.



$$(a, b) \longleftrightarrow \frac{a}{b}$$

Can we enumerate $\mathbb{Q}^+$?

$$\mathbb{Q}^+ = \left\{ \frac{a}{b} \;\middle|\; a,b \in \mathbb{Z}^+,\ a,b \text{ don't have common factors} \right\}$$

Follow the blue arrows again but <u>skip</u> any duplicates, call the enumeration s.

Thus we can enumerate $\mathbb{Q}$,

$$(0, s_1, -s_1, s_2, -s_2, s_3, -s_3, \ldots)$$

Are <u>all</u> infinite sets countable ??

<u>Fact</u> $\mathbb{R}$ is not countable!

Cantor's diagonalization argument.

$\aleph_0$ = cardinality of $\mathbb{Z}^+$

c = continuum = cardinality of $\mathbb{R}$

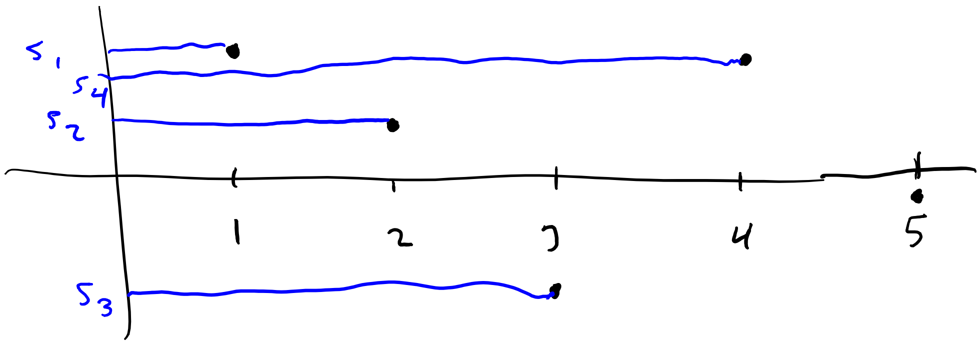
$$|\mathcal{P}(\mathbb{Z}^+)| = |\{S \subseteq \mathbb{Z}^+\}| \overset{?}{=} c$$

continuum hypothesis

There are consistent models of set theory which permit either answer!

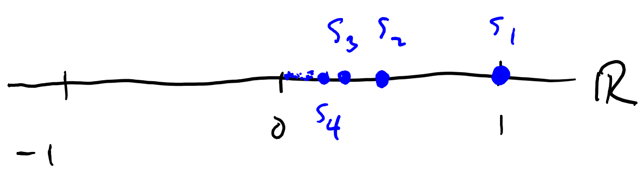
Visualizizations of sequences

If $s : \mathbb{Z}^+ \longrightarrow \mathbb{R}$, we can graph s:

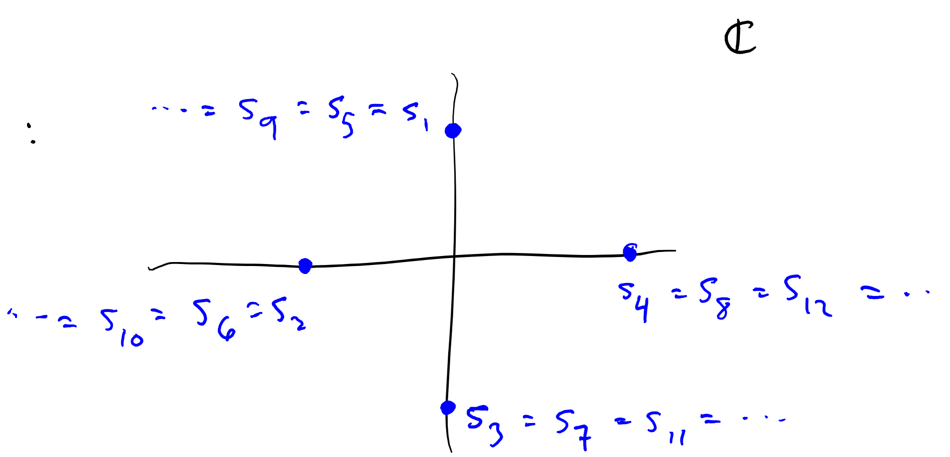


We can also plot our sequence:

$(s_n) = \left(\frac{1}{n}\right)$:

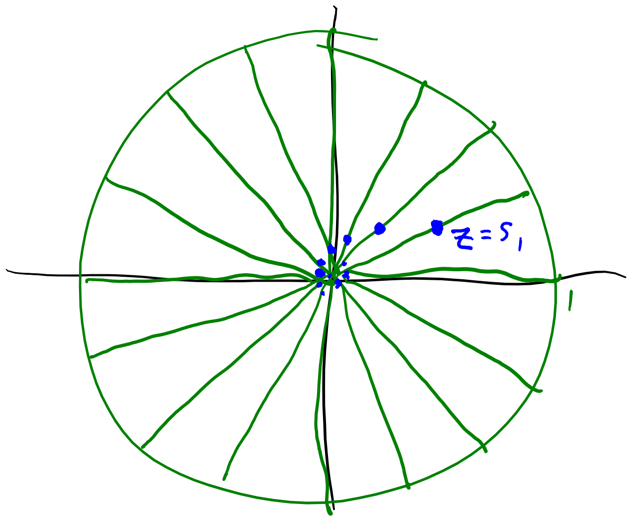


(i^n):



$z = z\left(\frac{1}{2}, \pi/8\right)$

$s = (z^n)$



spirals towards 0.