<u>EVT</u> Let $A \subseteq \mathbb{C}$ be compact, and let $f: A \to \mathbb{R}$ be cts.
Then $\exists\, l, u \in A$ s.t. $\forall x \in A$, $f(l) \le f(x) \le f(u)$.
I.e. f achieves max at u, min at l.

<u>Heine-Borel Theorem</u> $A \subseteq \mathbb{C}$ compact. For each $c \in A$ let $\delta_c > 0$.
Then $\exists$ finite subset $S \subseteq A$ s.t. $A \subseteq \bigcup_{c \in S} B(c, \delta_c)$.

<u>Pf of EVT assuming H-BT</u> Since f cts, $\forall a \in A\ \exists\, \delta_a > 0$
s.t. $\forall x \in A \cap B(a, \delta_a)$, $|f(x) - f(a)| < 1$. By H-BT,
$\exists$ finite $S \subseteq A$ s.t. $A \subseteq \bigcup_{a \in S} B(a, \delta_a)$.
Now $\{ |f(a)| + 1 \mid a \in S \}$ is a finite $\subseteq \mathbb{R}^+ \Rightarrow$ has an upper bd M.
If $x \in A$, then $x \in B(a, \delta_a)$ for some $a \in S \Rightarrow$

$$|f(x)| \le |f(x) - f(a)| + |f(a)| < 1 + |f(a)| \le M$$

Thus $\forall x \in A$, $f(x) \in [-M, M] \Rightarrow f(A)$ is a bdd subset of $\mathbb{R}$
$\Rightarrow \exists\, L = \inf f(A),\ U = \sup f(A) \in \mathbb{R}$.

Assume for $\otimes$ that $\forall a \in A$, $f(a) \ne L$. Then $\forall a \in A$, $f(a) - L > 0$
$\Rightarrow \exists\, \delta_a' > 0$ s.t. $x \in A \cap B(a, \delta_a') \Rightarrow |f(x) - f(a)| < \frac{f(a) - L}{2}$.
By HBT $\exists$ finite $S \subseteq A$ s.t. $A \subseteq \bigcup_{a \in S} B(a, \delta_a')$.
Take $K = \min\{ (f(a) + L)/2 \mid a \in S \} \in \mathbb{R}$.

Since $\forall a \in A.\ f(a) > L$, we learn that $K > L$. But

$$f(x) = f(x) - f(a) + f(a)$$

$$\geq -\frac{f(a) - L}{2} + f(a) \qquad [\text{def'n of } \delta_a' + x \in B(a, \delta_a') \text{ for some } a + f(a) > L]$$

$$= \frac{f(a) + L}{2}$$

$$\geq K$$

Thus K is a lower bd of $f(A) \Rightarrow K \leq L$. But $K > L$, so ⚡.

We learn that $\exists \ell \in A$ s.t. $L = f(\ell)$.

Sim arg for $U = f(u)$. $\square$

Derivatives Read Chap. 6 and refamiliarize yourself w/ derivative thms.

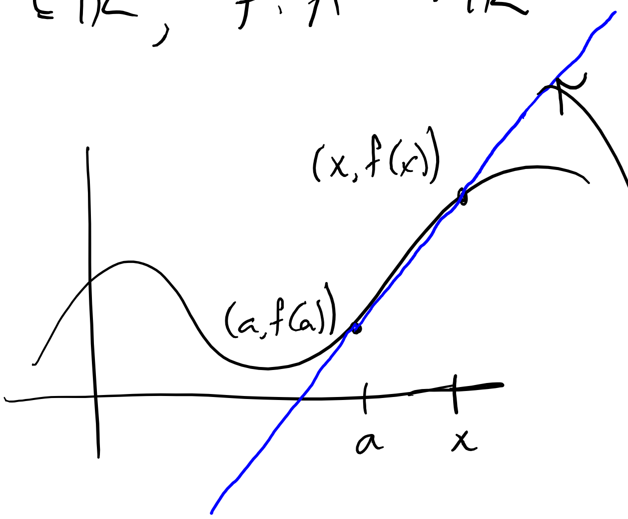
For $A \subseteq \mathbb{C}$, $a \in A$, $f : A \to \mathbb{C}$ is *differentiable* at a if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

If f is diff'l at a, we call the above limit the *derivative* of f at a, denote $f'(a)$ or $\left.\frac{df}{dx}\right|_{x=a}$.

All of our derivative theorems go through:

- power rule
- linearity of $(\ )'$: $(f+g)' = f' + g'$
  $c \in \mathbb{C}$, $(c \cdot f)' = c \cdot (f')$
- product & quotient rules
- chain rule.

If $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$

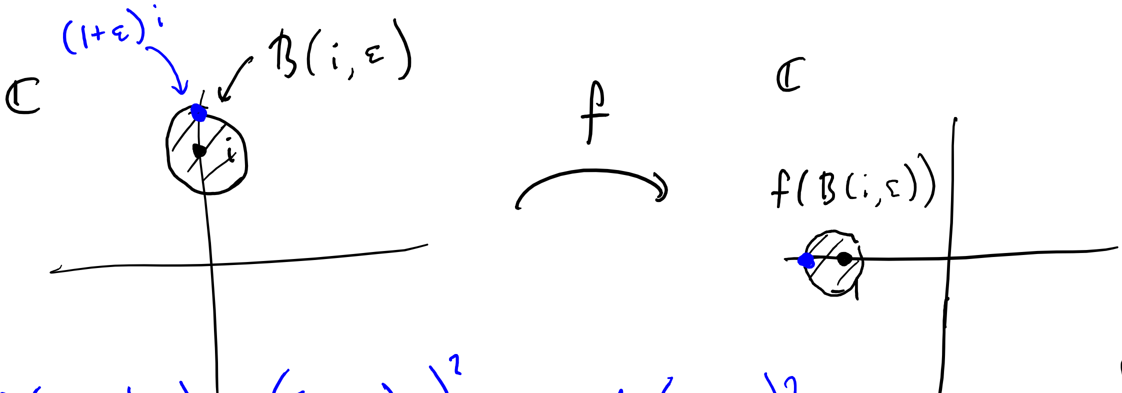


$\dfrac{f(x) - f(a)}{x - a}$ is the slope of

If $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, what does $f'(a)$ "mean"?

Consider $f(z) = z^2$ for $z \in \mathbb{C}$.

Then $f'(z) = 2z$. Thus $f'(i) = 2i$.



$$f((1+\varepsilon)i) = ((1+\varepsilon)i)^2 = -1(1+\varepsilon)^2$$

$$\frac{f((1+\varepsilon)i) - f(i)}{(1+\varepsilon)i - i} = \frac{-(1+\varepsilon)^2 - (-1)}{\varepsilon i}$$

$$= \frac{-2\varepsilon - \varepsilon^2}{\varepsilon i} = \frac{-2-\varepsilon}{i}$$

$$= (-2-\varepsilon)(-i)$$

$$= 2i + \varepsilon i$$

$$\Big\downarrow \varepsilon \to 0$$

$$2i$$

Friday, March 20, 2015 8:40 AM

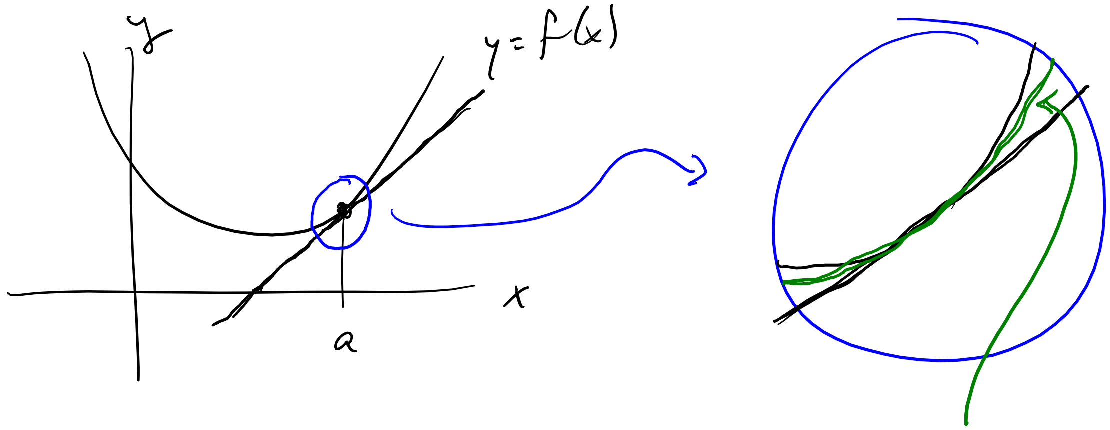
Linear approximation Assume f has cts derivative close to a

$$f(x) = f(x) - f(a) + f(a)$$

$$= \frac{f(x)-f(a)}{x-a} \cdot (x-a) + f(a) \quad [\text{if } x \neq a]$$

$$\approx f'(a) \cdot (x-a) + f(a) \quad \text{for } x \text{ near } a.$$

Q Can we do better than just linear approx?



Is there a quadratic that does better??

Define $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$ whenever $f^{(n-1)}(x)$ is diff'l at x where

$$f^{(0)}(x) = f(x), \quad f^{(1)}(x) = \left(f^{(0)}(x)\right)' = f'(x), \ldots$$

<u>Defn</u> Let f be a fn w/ n-th order derivatives at $x=a$ in the domain of f. Then the <u>n-th order Taylor polynomial</u> of f at $x=a$ is

$$T_{n,f,a}(x) = f(a) + f'(a)\cdot(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\cdot(x-a)^k$$