Lecture 3

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<u>Proof</u> Let x, y be real numbers. ~~We know~~ ~~$x \le |x|$ & $y \le |y|$. So by the earlier claim,~~ We know

$$-|x| \le x \le |x|$$

and $$-|y| \le y \le |y|$$

Taking the sum,

$$-|x| - |y| \le x + y \le |x| + |y|$$

Rewriting,

$$-(|x| + |y|) \le x + y \le |x| + |y|.$$

By the claim, we learn that

$$|x+y| \le |x| + |y|$$

as desired. $\square$

<u>Claim</u> If $|x-5|<4$, then $|x^3-3x|<1030$

<u>**The Big Rule**</u> Only assume the assumptions in proofs.

<u>Pf</u> Assume $|x-5|<4$. Then

$$
\begin{aligned}
|x^3-3x| &\le |x^3| + |-3x| \qquad (\triangle \text{ ineq.}) \\
&= |x|^3 + 3|x| \\
&= |x-5+5|^3 + 3|x-5+5| \\
&\le (|x-5|+5)^3 + 3(|x-5|+5) \qquad (\triangle \text{ ineq}) \\
&< (4+5)^3 + 3(4+5) \qquad (-)^3 \text{ is increasing (by assumption)} \\
&< 10^3 + 3\cdot 10 \\
&= 1030. \qquad \square
\end{aligned}
$$

Proof by contradiction

~~To prove $P \Rightarrow Q$ we can assume $\neg Q$~~
~~and show that when we assume~~

To prove $P \Rightarrow Q$ assume P & $\neg Q$ and show that something impossible/absurd happens.

Good: powerful & easy to use

Bad: inelegant & non-constructive

Quantifiers

Universal quantifier: For every x of a particular type

Existential quantifier: There exists x w/ particular qualities

Universal quantifiers

Shorthand:

"$\forall x.\ P(x)$" = "For every x (of a particular type), $P(x)$ is true."

property depending on x

e.g. For every $x \geq 1$, $x > 0$.

= For all $x \geq 1$, $x > 0$

= $\forall x \geq 1.\ x > 0$

e.g. Every positive integer is a product of prime numbers.

= $\forall x$ a positive integer. x is a product of primes.

Break this down further w/ an existential quatifier

= $\forall x.\ \exists$ primes $p_1, p_2, \ldots, p_n$ such that $x = p_1 \cdot p_2 \cdots p_n$

there exists

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More existential statements:

- There exist real numbers which are not rational

  $\exists\, x$ a real number . x is not rational

  such that

- There exist fns which are continuous everywhere and differentiable nowhere

  $\exists\, f$ a fn. f is cts but diff'l nowhere

  $\exists\, f$ a fn. $\forall\, a$ a real # . f is cts at a and f is not diff'l at a

Switching the order of quantifiers drastically changes meaning.

Weierstrass devil function — later.

Our favorite combo of univ & exist'l quantifiers:

$$\forall \varepsilon > 0 .\ \exists \delta > 0 .\ (0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

Defines $\lim_{x \to a} f(x) = L$

How do we prove quantified statements?

$\forall x . P(x)$ —— show that $P(x)$ holds for every x of given type

"Let x be of the given type. ..."

$\exists x . P(x)$ —— produce (at least) one x of given type such that $P(x)$ is true.

Q What if there are no x of the specified type?