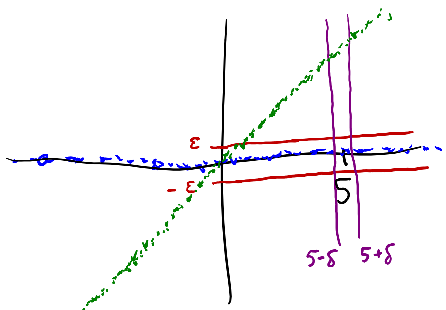
Lecture 29

Tuesday, March 17, 2015 8:02 AM

$I : \mathbb{R} \longrightarrow \mathbb{R}$

$$I(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Where is I continuous? A At $x = 0$



$I(5) = 0$

Is $\lim_{x \to 5} I(x) = I(5) = 0$?

We can never get only rational #s b/w $5-\delta$ and $5+\delta$, thus I is not cts at 5.

Similarly, I is not cts away from 0.

But $\lim_{x \to 0} I(x) = 0 = I(0)$.

Pf Given $\varepsilon > 0$ set $\delta = \varepsilon > 0$ and assume

(*) $0 < |x - 0| < \delta = \varepsilon$. Then

$$|I(x) - 0| = \begin{cases} |0| & \text{if } x \in \mathbb{Q} & < \varepsilon \\ |x| & \text{if } x \in \mathbb{R} - \mathbb{Q} & < \varepsilon \text{ by (*)}. \end{cases}$$

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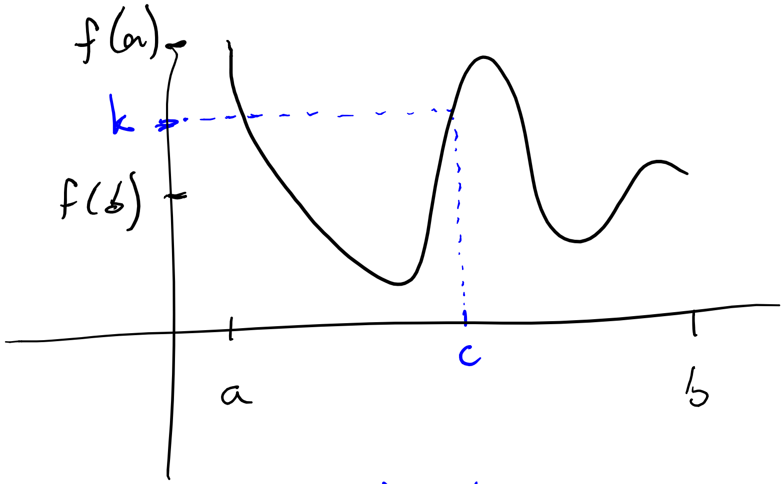
Thus $|I(x)| < \varepsilon$ whenever $0 < |x| < \delta = \varepsilon$, so

$\lim_{x \to 0} I(x) = 0$. $\square$

IVT (Intermediate value theorem)

f cts



$f(c) = k$

Thm

For $a, b \in \mathbb{R}$, $a < b$, $f : [a,b] \to \mathbb{R}$ continuous, let $k \in \mathbb{R}$ be strictly between $f(a)$ & $f(b)$. Then $\exists\, c \in (a,b)$ s.t. $f(c) = k$.

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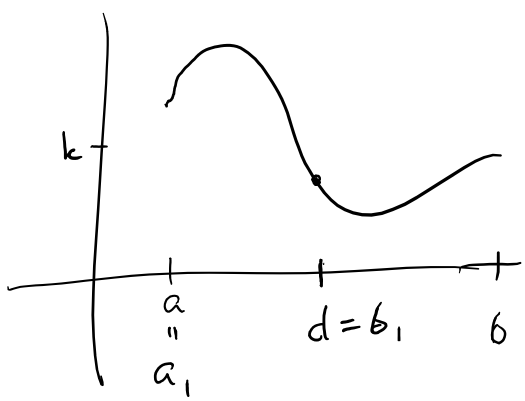
Pf Set $a_0 = a$, $b_0 = b$. Given $a_{n-1}, b_{n-1} \in [a,b]$ with k strictly b/w $f(a_{n-1})$ & $f(b_{n-1})$, let

$$d = \frac{a_{n-1} + b_{n-1}}{2}.$$

If $f(d) = k$, then we're done.

If $f(d) \neq k$, then either $f(d) < k$ or $f(d) > k$.

Let $e = a_{n-1}$ or b_{n-1} so that $f(e) - k$ and $f(d) - k$ have opposite signs.



$f(d) - k < 0$ so set $e = a$

Then k is strictly between $f(d)$ & $f(e)$.

Set the smaller of d, e to be a_n; the larger to be b_n

Observe that $a_{n-1} \le a_n < b_n \le b_{n-1}$, and $|b_n - a_n| = \frac{1}{2}|b_{n-1} - a_{n-1}|$

Either this process terminates in finitely many steps <u>or</u> we get $[a,b]=[a_0,b_0]\supseteq[a_1,b_1]\supseteq[a_2,b_2]\supseteq\cdots$ w/ $b_n-a_n=\frac{1}{2^n}(b-a)$ and k is always strictly b/w $f(a_n)$ & $f(b_n)$. Thus

$$a=a_0\leq a_1\leq a_2\leq\cdots\leq a_n\leq\cdots\leq b_n\leq\cdots\leq b_2\leq b_1\leq b_0=b$$

Set $c=\sup\{a_0,a_1,a_2,\dots\}$ (which exists b/c $\{a_0,a_1,\dots\}$ is bdd above [say by b] and $\mathbb{R}$ is complete).

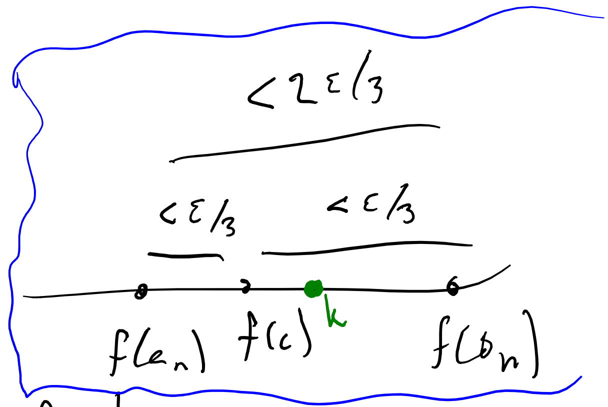
<u>Claim</u> $f(c)=k$.

Let $\varepsilon>0$. Since f is cts, it is cts at c, so $\exists\delta>0$ s.t. $\forall x\in[a,b]$, if $|x-c|<\delta$, then $\left|f(x)-f(c)\right|<\varepsilon/3$. Take $n\in\mathbb{Z}^+$ so that $\frac{1}{2^n}<\frac{\delta}{b-a}$. Then $|a_n-c|\leq|a_n-b_n|=\frac{b-a}{2^n}<\delta$.

So $|f(a_n) - f(c)| < \varepsilon/3$. Similarly,

$|f(b_n) - f(c)| < \varepsilon/3$. Thus

$$
\begin{aligned}
|f(b_n) - f(a_n)| &= |f(b_n) - f(c) + f(c) - f(a_n)| \\
&\le |f(b_n) - f(c)| + |f(a_n) - f(c)| \\
&< \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3.
\end{aligned}
$$



Since k is strictly between $f(a_n)$ & $f(b_n)$

$$|f(a_n) - k|,\ |f(b_n) - k| < 2\varepsilon/3.$$

Now $|f(c) - k| \le |f(c) - f(a_n)| + |f(a_n) - k|$

$$< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Since ε was an arbitrary positive #, we learn that

$f(c) = k$. $\square$

<u>Application</u> Every odd degree real polynomial has a root in $\mathbb{R}$.

<u>Pf</u> Suppose $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ with $a_i \in \mathbb{R}$, $a_n \neq 0$, n odd.

Then $f : \mathbb{R} \to \mathbb{R}$ is cts. By IVT, to get $c \in \mathbb{R}$ s.t. $f(c) = 0$, it suffices to find $a, b \in \mathbb{R}$ s.t. $f(a) < 0$, $f(b) > 0$.

<u>Check</u> $\lim\limits_{x \to \infty} \dfrac{f(x)}{a_n x^n} = 1 = \lim\limits_{x \to -\infty} \dfrac{f(x)}{a_n x^n}$.

Thus for $|x|$ sufficiently large, $f(x)$ & $a_n x^n$ have the same sign. Since $a_n x^n$ has opposite signs for $x > 0$, $x < 0$, we get that $f(x)$ switches signs as well. $\square$