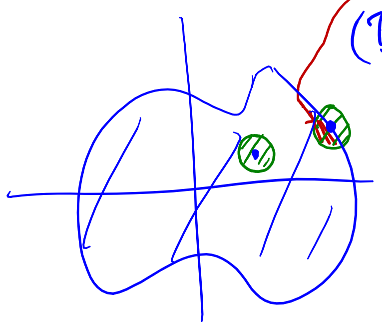
Compact set in $\mathbb{R}$, or $\mathbb{C}$, is a closed bounded subset.

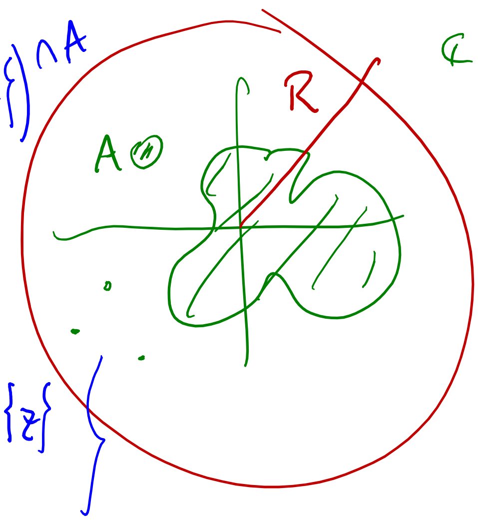
$A \subseteq \mathbb{C}$ is bdd if $\exists R > 0$ s.t. $\forall a \in A$, $|a| \leq R$

$A \subseteq \mathbb{C}$ is closed

if • $\underbrace{\partial A}_{\text{boundary of } A} \subseteq A$

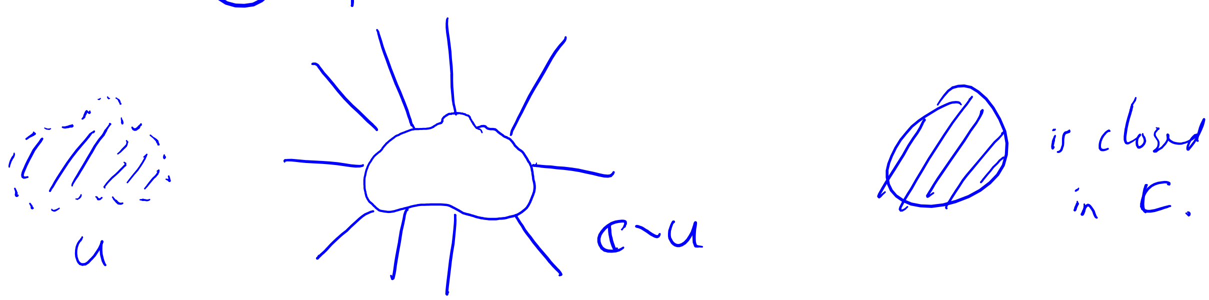
$$\partial A = \left\{ z \in \mathbb{C} \;\middle|\; \forall \delta > 0 \;\; (B(z,\delta) \setminus \{z\}) \cap A \neq \emptyset \text{ and } (B(z,\delta)\setminus\{z\}) \cap A \neq B(z,\delta)\setminus\{z\} \right\}$$





• equivalently, A is the complement in $\mathbb{C}$ of an open set $A = \mathbb{C} \setminus U$, $U \subseteq \mathbb{C}$ open.

I.e. $U = \bigcup$ open balls.



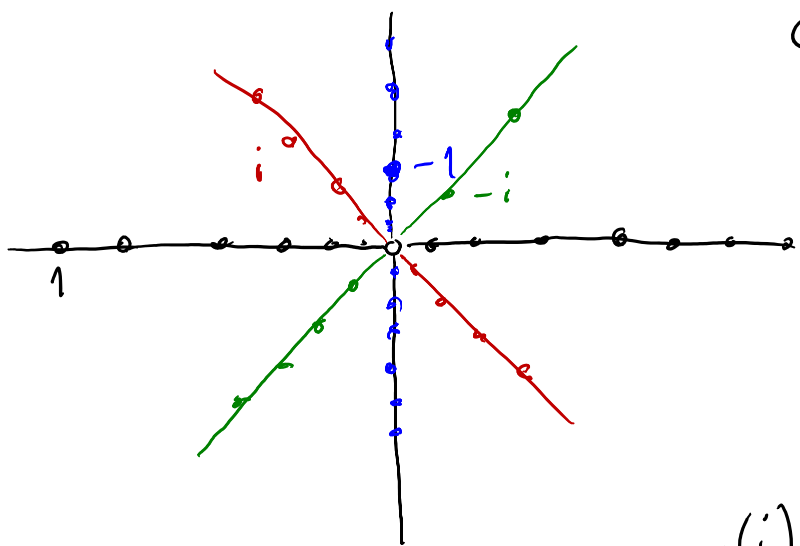
Example Look at $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{\bar{x}}{x}$

$= 1$ for $x \neq 0$

Then $\lim_{x \to 0} f(x) = 1$. — take $\delta = 1$ or $1,000,000$ in ε-δ proof and it works!

$g: \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $g(z) = \frac{\bar{z}}{z}$

$\mathbb{C} \setminus \{0\}$



$g(i) = \frac{\bar{i}}{i} = \frac{-i}{i}$

$= -1$

$g(1+i) = \frac{1-i}{1+i} = -i$

$\lim_{z \to 0} g(z)$ does not exist!

$g(6i) = \frac{\overline{6i}}{6i} = \frac{-6i}{6i}$

$= -1$

Complex fns have limits determined only by approaching from <u>all directions</u>!

Continuity $A, B \subseteq \mathbb{C}$

Def'n A function $f: A \to B$ is continuous at $a \in A$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall x \in A$, if

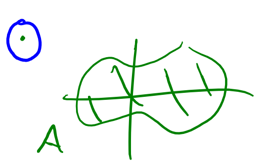
$$|x-a| < \delta, \text{ then } |f(x) - f(a)| < \varepsilon.$$

The function f is continuous everywhere if it is cts at every $a \in A$.

Thm For $f: A \to B$, $a \in A$

1. If a is not a limit point of A, then f is continuous at a.
2. If a is a limit point of A, then f is cts at a iff $\lim_{x \to a} f(x) = f(a)$.

Recall $a \in \mathbb{C}$ is a limit point of $A \subseteq \mathbb{C}$ if $\forall \varepsilon > 0$, $(B(a, \varepsilon) \setminus \{a\}) \cap A \neq \emptyset$.



not a limit pt of A

Pf ① Assume $a \in A$ is <u>lonely</u>. Then $\exists \delta > 0$ s.t. $B(a, \delta) \cap A = \{a\}$. Given $\varepsilon > 0$, take δ as above. Then the only $x \in A$ s.t. $|x-a| < \delta$ is $x = a$. Thus if

$$|x-a| < \delta, \text{ then } |f(x) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon.$$

Thus f is cts at our lonely point a.

② Moral exercise: (exactly the limit condition except we're allowing $|x-a| = 0$.) $\square$

<u>Thm</u> [stated informally] Constant fns, linear fns, the absolute value fn, polynomials, rational fns (where denominator is $\neq 0$), $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ for f cts, composites of cts fns, scalar multiples of cts fns, sums and differences of cts fns, products of cts fns, quotients of cts fns $\frac{f}{g}$ for x s.t. $g(x) \neq 0$, powers of cts fns ARE ALL CONTINUOUS.

Pf Follows immediately from previous thm and limit thms! $\square$

e.g.

$$I(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$I : \mathbb{R} \to \mathbb{R}$ Where is I cts?