Lecture 26

Saturday, March 7, 2015 8:33 PM

Limit theorems

Thm If $\lim_{x \to a} f(x)$ exists, then it is unique.

Pf Suppose L_1, L_2 are limits of f as $x \to a$. Assume for contradiction $L_1 \neq L_2$, in which case $|L_1 - L_2| > 0$. Then $\exists \delta_1, \delta_2 > 0$ s.t. $0 < |x-a| < \delta_i \Rightarrow |f(x) - L_i| < \frac{|L_1 - L_2|}{2}$.

[We are invoking the def'n of limit (twice) with $\varepsilon = \frac{|L_1-L_2|}{2}$.]

Set $\delta = \min\{\delta_1, \delta_2\}$ and suppose $0 < |x-a| < \delta$. Then

$$
\begin{aligned}
|L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\
&\leq |L_1 - f(x)| + |f(x) - L_2| && [\Delta \text{ ineq}] \\
&< \frac{|L_1 - L_2|}{2} + \frac{|L_1 - L_2|}{2} && [\text{since } \delta \leq \delta_1, \delta_2] \\
&= |L_1 - L_2|
\end{aligned}
$$

Thus $|L_1 - L_2| < |L_1 - L_2|$, $\otimes$. Thus $L_1 = L_2$. $\square$

Thm Suppose $\lim_{x \to a} f(x) = L \neq 0$. Then $\exists \delta > 0$ s.t. if $0 < |x-a| < \delta$, then $|f(x)| > \frac{|L|}{2}$.

pf For $\varepsilon = \frac{|L|}{2} > 0$, $\exists \delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \frac{|L|}{2}$.

Thus $\frac{|L|}{2} > |f(x) - L| \geq |L| - |f(x)| \Rightarrow |f(x)| > \frac{|L|}{2}$ for $0 < |x-a| < \delta$.

$\square$

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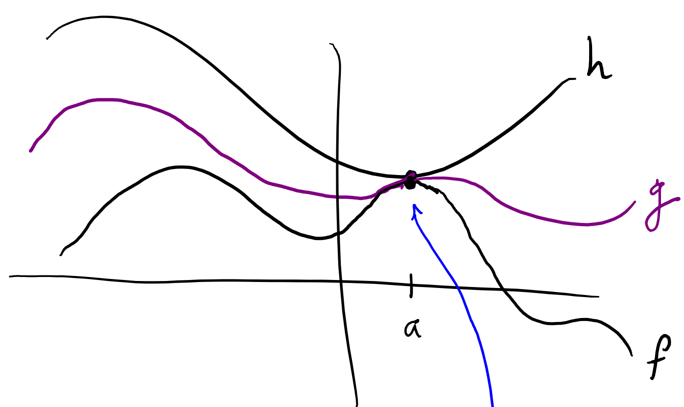
[I can offer nothing better than pp. 157 – 161 of the book in terms of proofs and explanations of the limit theorems. Read and understand them.]

Here, though, is a picture of the Squeeze Theorem:

Suppose $f, g, h : A \to \mathbb{R}$ and $\forall x \in A \setminus \{a\}$, $f(x) \leq g(x) \leq h(x)$.
If $\lim_{x \to a} f(x)$ & $\lim_{x \to a} h(x)$ exist and are equal, then $\lim_{x \to a} g(x)$ exists and $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x)$.



Codomain is $\mathbb{R}$ (not $\mathbb{C}$).