Lecture 25

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Limits & non-limits

Def'n Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$. The limit of $f(x)$ as x approaches a is L if $\forall$ real $\varepsilon > 0$ $\exists$ real $\delta > 0$ s.t. for all $x \in A$, if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

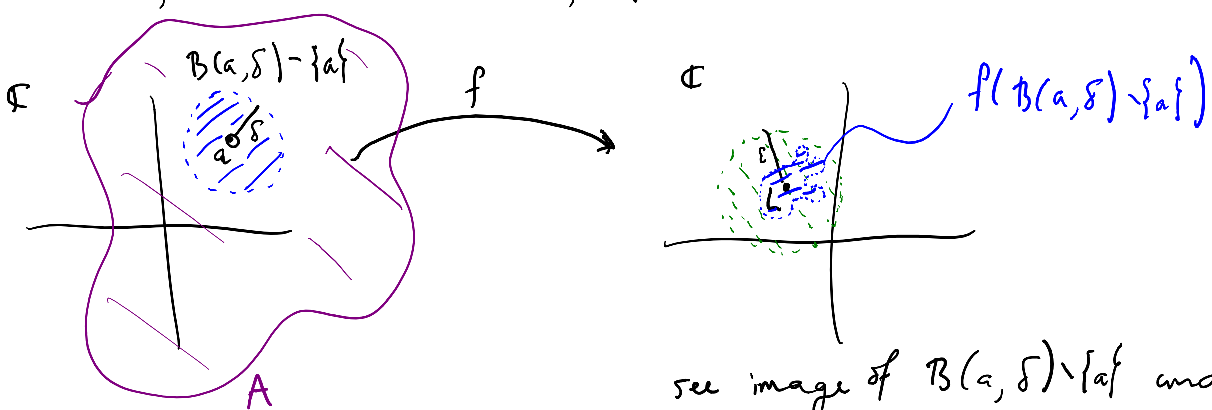
I.e. $\underbrace{\forall \varepsilon > 0}$ $\underbrace{\exists \delta > 0}$ s.t. $f\Big(\underbrace{(B(a,\delta) \cap A) \setminus \{a\}}_{\text{this is the set of } x \in A \text{ such that } 0<|x-a|<\delta}\Big) \subseteq \underbrace{B(L, \varepsilon)}_{\text{the set of } y \in \mathbb{C} \text{ s.t. } |y-L|<\varepsilon}$

No matter which $\varepsilon > 0$ we're given

we can find $\delta > 0$ s.t. when we

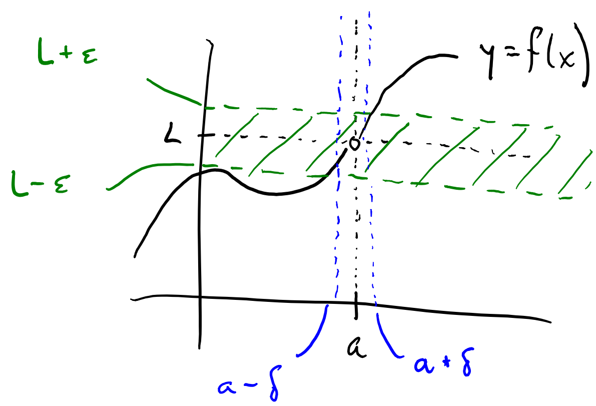
apply f to this set and it lands inside of $B(L, \varepsilon)$

Given $\varepsilon > 0$, we can find $\delta > 0$, s.t.



see image of $B(a,\delta) \setminus \{a\}$ under f landing inside of $B(L, \varepsilon)$.

If A, image$(f) \subseteq \mathbb{R}$, we get the familiar picture:



Given $\varepsilon > 0$, we can find $\delta > 0$ s.t. when x is within δ of a (but $x \neq a$), $f(x)$ is within ε of L.

If L is the limit of $f(x)$ as x approaches a, write

$$L = \lim_{x \to a} f(x) \quad \underline{\text{or}} \quad f(x) \xrightarrow[x \to a]{} L.$$

<u>e.g.</u> $\displaystyle\lim_{x \to 2} (2x+2) = 6$.

<u>Pf</u> Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{2}$ (which is a positive real #) and assume $0 < |x-2| < \delta = \frac{\varepsilon}{2}$. Then

$$\begin{aligned} |2x+2-6| &= |2x-4| \\ &= 2|x-2| \\ &< 2 \cdot \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus we have shown that for any $\varepsilon > 0$, $\exists\, \delta \left(= \frac{\varepsilon}{2}\right) > 0$ s.t. if $0 < |x-2| < \delta$, then $|2x+2-6| < \varepsilon$, i.e. $\displaystyle\lim_{x \to 2} (2x+2) = 6$.

Q Where the heck did that come from?!

The steps are clear enough, but why in the world would we start with $\delta = \frac{\varepsilon}{2}$?

A Our "ε-δ proof" is a final step in a longer process of discovery. It certifies that the result is true w/out revealing where it came from. That's fine, but we will have to <u>invent</u> ε-δ proofs, so let's look into their origins:

Target: $|f(x) - L| < \varepsilon$

$$|(2x+2) - 6| = |2x - 4|$$

We can control: $|x - a| = |x - 2|$ w/ δ.

δ can depend on ε, but it can't depend on x.

We seek $\delta = \delta(\varepsilon)$ so that $0 < |x-2| < \delta$ implies $|2x-4| < \varepsilon$.

Question: What is the relationship b/w $|x-2|$ and $|2x-4|$?

Aha! $|2x-4| = 2|x-2|$.

So we want $|2x-4| = 2|x-2| < \varepsilon$. Dividing by 2, we see that this happens if $\underbrace{|x-2|}_{\text{exactly what we have control over!}} < \frac{\varepsilon}{2}$

So set $\delta = \frac{\varepsilon}{2}$! The ε-δ proof essentially reverses these steps in order to certify that the chosen δ works.

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Important Read the notes for many more examples.

(Alas — it gets more complicated!)

Non-limits

Q1 What does $L \neq \lim_{x \to a} f(x)$ mean?

Q2 What does it mean for $\lim_{x \to a} f(x)$ to not exist?

$\lim_{x \to a} f(x)$ does not exist if $\forall L$, $L \neq \lim_{x \to a} f(x)$.

Thus Q2 reduces to Q1:

$L \neq \lim_{x \to a} f(x)$ is the negation of $L = \lim_{x \to a} f(x)$.

$L = \lim_{x \to a} f(x)$: $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\left(0 < |x-a| < \delta\right) \Rightarrow \left(|f(x) - L| < \varepsilon\right)$ ($x \in A$ &)

$L \neq \lim_{x \to a} f(x)$: $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\neg\left(\left(0 < |x-a| < \delta\right) \Rightarrow \left(|f(x) - L| < \varepsilon\right)\right)$ ($x \in A$ &)

Recall $\neg(P \Rightarrow Q)$ is equiv to $P \wedge (\neg Q)$

$\neg\left(|f(x) - L| < \varepsilon\right)$ is $|f(x) - L| \geq \varepsilon$.

Thus

$$\boxed{L \neq \lim_{x \to a} f(x): \ \exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0\ \exists x \in A \text{ s.t. } 0 < |x-a| < \delta \text{ and } |f(x) - L| \geq \varepsilon.}$$

e.g. Let $f: \mathbb{C}\setminus\{0\} \to \mathbb{C}$. $\forall L \in \mathbb{C}$, $L \neq \lim_{x\to 0} f(x)$.
$x \mapsto \frac{x}{|x|}$

Pf Let $\varepsilon = 1$ and suppose δ is some positive real #. Let $x = -\frac{\delta}{2}$ if $Re(L) \geq 0$, $x = \frac{\delta}{2}$ if $Re(L) < 0$. Then $x \in \mathbb{C}-\{0\}$ and $0 < |x-0| < \delta$ b/c $|x-0| = |\pm\delta/2| = \delta/2$. Observe, then, that

$$|f(x) - L| = \begin{cases} |-1-L| & \text{if } Re(L) \geq 0 \\ |1-L| & \text{if } Re(L) < 0 \end{cases} \geq 1.$$

Thus $\lim_{x\to 0} \frac{x}{|x|} \neq L$. $\square$