Lecture 24

Saturday, March 7, 2015 5:54 PM

Topology & density: $\mathbb{R}$ & $\mathbb{C}$

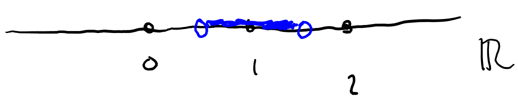
Both $\mathbb{R}$ & $\mathbb{C}$ have an absolute value $|\ |$ which then gives distance: $|x-y|$ is the distance b/w x & y.

This allows us to make some unified definitions for each of $\mathbb{R}$, $\mathbb{C}$:

Defn Let $F = \mathbb{R}$ or $\mathbb{C}$. Then the open ball centered at $a \in F$ of radius $r \in \mathbb{R}_{\geq 0}$ is

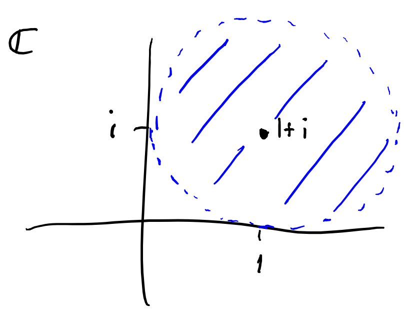
$$B_F(a,r) = B(a,r) = \{ x \in F \mid |x-a| < r \}.$$

$$\text{So } B_{\mathbb{R}}(1, 1/2) = \{ x \in \mathbb{R} \mid \underbrace{|x-1| < \tfrac{1}{2}}_{\text{distance less than } 1/2 \text{ from } 1} \}$$



I.e. $B_{\mathbb{R}}(1, 1/2) = (1/2, 3/2)$.

$$B_{\mathbb{C}}(1+i, 1) = \{ x \in \mathbb{C} \mid \underbrace{|x-(1+i)| < 1}_{\text{distance less than } 1 \text{ from } 1+i} \}$$



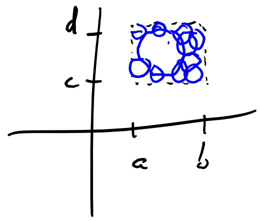
An <u>open set</u> in F is a (possibly infinite) union of open balls.

<u>e.g.</u> • In $\mathbb{R}$, $(a,b) = B\left((a+b)/_2, \, |a-b|/_2\right)$ is always open

• $\emptyset$ is open b/c its the empty union!

• $(a,b) \times (c,d)i \subseteq \mathbb{C}$ is open, but we have to use infinitely many balls!

<u>Challenge</u> Find an explicit collection of balls whose union is $(a,b) \times (c,d)i$.



<u>Prop</u> Every nonempty open set contains infinitely many points.

<u>Pf</u> For each integer $n \geq 2$, $a + r/n \in B(a,r)$. $\square$

<u>Cor</u> $\{a\} \subseteq F$ is <u>not</u> open.

<u>Thm</u> • $\emptyset$ & F are open

• arbitrary unions of opens are open

• finite intersections of opens are open

i.e. intersections of finitely many

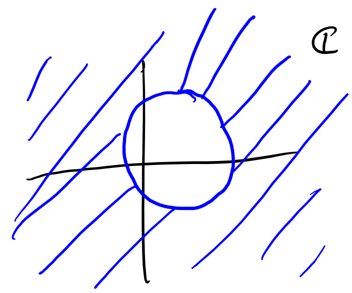
<u>Aside</u> This means that F with its open subsets is a <u>topological space</u>.

<u>Pf</u> Reading. $\square$

But note: $\bigcap_{n \in \mathbb{Z}^+} B(a, 1/n) = \{a\}$, so <u>finite</u> intersections is essential!

<u>Def'n</u> A set $A \subseteq F$ is <u>closed</u> if it is the complement of an open set $U \subseteq F$: $A = F \setminus U$.

<u>e.g.</u> • $F \setminus B(a,r)$ is closed : 



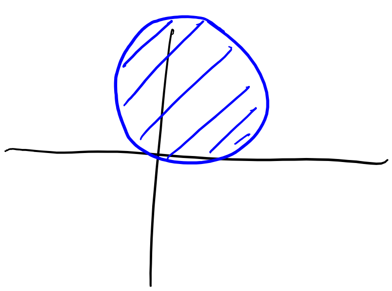
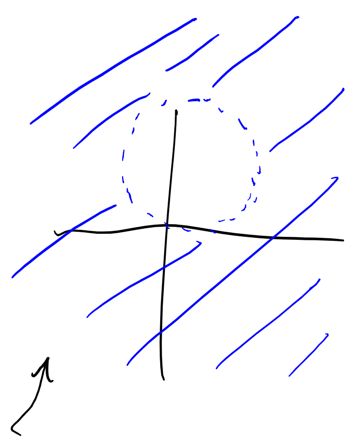
• $D(a,r) = \{ x \in F \mid |x-a| \leq r \}$ is closed

<u>disk</u> of radius r centered at a.

 $= \mathbb{R} \setminus$ 

$(b, \infty) = \bigcup_{n=1}^{\infty} B(a,1)$

open

 $= \mathbb{C} \setminus$ 

open by sim argument to 

<u>Def'n</u> The <u>closure</u> of a set $A \subseteq F$, denoted $\bar{A}$, is the smallest closed subset of F containing A.

<u>e.g.</u> $\overline{B(a,r)} = D(a,r)$

<u>Def'n</u> $A \subseteq F$ is <u>dense</u> if $\bar{A} = F$.

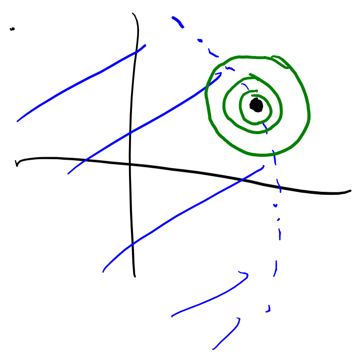
More generally, if $S \subseteq F$ is closed, say $A \subseteq S$ is dense in S if $\bar{A} = S$.

<u>Thm</u> $\mathbb{Q} \subseteq \mathbb{R}$ is dense in $\mathbb{R}$.

<u>Def'n</u> Let $A \subseteq F$. A point a is in the <u>interior</u> of A if $\exists r > 0$ s.t. $B(a,r) \subseteq A$. A point a is on the <u>boundary</u> of A if $\forall r > 0$, $B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap (F \setminus A) \neq \emptyset$.

$\mathrm{Int}(A) = \{\text{interior pts of } A\}$

$\mathrm{Bd}(A) = \{\text{boundary pts of } A\}$.



<u>Thm</u> $\forall A \subseteq F$, $\bar{A} = A \cup \mathrm{Bd}(A)$

<u>Pf</u> Reading.

→ So this says $\bar{\mathbb{Q}} = \mathbb{R} \Rightarrow \forall x \in \mathbb{R} \setminus \mathbb{Q}$, $B(x,r) \cap \mathbb{Q} \neq \emptyset$, and $B(x,r) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$ for all $r > 0$.

In particular, $\forall r > 0$ and any $x \in \mathbb{R}$, there is always a rational number within r of x