Lecture 21

Monday, March 2, 2015 7:55 AM

The Complex Numbers

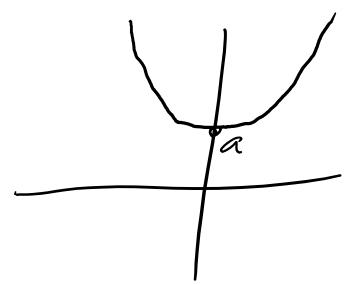
$\mathbb{R}$ is complete & ordered

$S \subseteq \mathbb{R}$ bdd above, then $\sup(S)$ exists and is an elt of $\mathbb{R}$

But

$\mathbb{R}$ lacks roots to many polynomials, e.g.

if $a > 0$, then $x^2 + a = 0$ has no sol'ns!



Idea of $\mathbb{C}$, the complex numbers:

build $\mathbb{C}$ from $\mathbb{R}$ so that it is a field containing sol'ns to $x^2 + a = 0$.

$x^2 = -a$

Starting point: $\mathbb{C} = \mathbb{R} \times \mathbb{R}$

Let the first coord act like $\mathbb{R}$

Use the second coord so that $(0, \sqrt{a})$ to solve $x^2 = -a$, i.e. $(0, \sqrt{a}) \cdot (0, \sqrt{a}) = (-a, 0)$

Let's set $(a,b)+(c,d) = (a+c, b+d)$.

What should $(a,b)\cdot(c,d)$ look like?

$$(a,b)\cdot(c,d) = \big((a,0)+(0,b)\big)\cdot\big((c,0)+(0,d)\big)$$

$$= (a,0)\cdot(c,0) + \underbrace{(a,0)\cdot(0,d)}$$

$$+ (0,b)\cdot(c,0) + \underbrace{(0,b)}_{\substack{\text{solves}\\ x^2=-b^2}}\cdot\underbrace{(0,d)}_{\substack{\text{solves}\\ y^2=-d^2}}$$

$$= (ac,0) + (bd,0)$$

$$+ (0,ad) + (0,bc)$$

$$= (ac \pm bd,\ ad+bc)$$

needs to be $-bd$ in order that the other axioms work out!

$(a,0)$ solves $x^2=a^2$

$(0,d)$ solves $y^2=-d^2$

$(xy)^2 = -(ad)^2$

$\Rightarrow (a,0)(0,d) = (0,ad)$

$\Downarrow$

$$(xy)^2 = x^2y^2 = (-b^2)(-d^2) = (bd)^2$$

$\Downarrow$

$(0,b)(0,d)$

$= \cancel{(bd,0)}$

$(\pm bd, 0)$

Conclusion $(a,b)\cdot(c,d)$

$$= (ac-bd,\ ad+bc).$$

We can then check that

1. $+, \cdot$ are comm assoc binary ops
2. $\cdot$ distributes over $+$
3. $(0,0)$ is the additive identity
4. $-(a,b) = (-a,-b)$
5. $(1,0)$ is the multiplicative identity

In order for $\mathbb{C}$ to be a field, we need mult inverses of nonzero elts:

Try to solve $(a,b)\cdot(c,d) = (1,0)$ for (c,d)

$$(ac-bd, ad+bc) = (1,0)$$

$$\iff \quad ac-bd = 1 \quad \& \quad ad+bc = 0$$

$$\frac{-a^2 d}{b} - bd = 1 \quad \longleftarrow \quad c = \frac{-ad}{b}$$

$$\left(-\frac{a^2}{b} - b\right) d = 1$$

$$-\frac{(a^2+b^2)}{b} d = 1 \implies d = \frac{-b}{a^2+b^2}$$

$$c = \frac{-ad}{b} \qquad c = \frac{ab}{b(a^2+b^2)} = \frac{a}{a^2+b^2}$$

$$d = \frac{-b}{a^2+b^2}$$

I.e. $$(a, b)^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \right)$$

! Steps were reversible/valid for $b \neq 0$.

If $b = 0$, $(a, 0)^{-1} = \left(\frac{1}{a}, 0 \right)$

$$= \left(\frac{a}{a^2+0^2}, \frac{-0}{a^2+0^2} \right) \checkmark$$

Thus $\mathbb{C}$ is a field.

Special way of writing elts of $\mathbb{C}$:

For real #s a, b, let $a + bi = (a, b)$.

Note then that $0 + 1 \cdot i = i = (0, 1)$

and $(0,1)$ solves $x^2 + 1 = 0$ i.e. $x^2 = -1$

thus $i^2 = -1$. Of course, $(-i)^2 = -1$ as well.

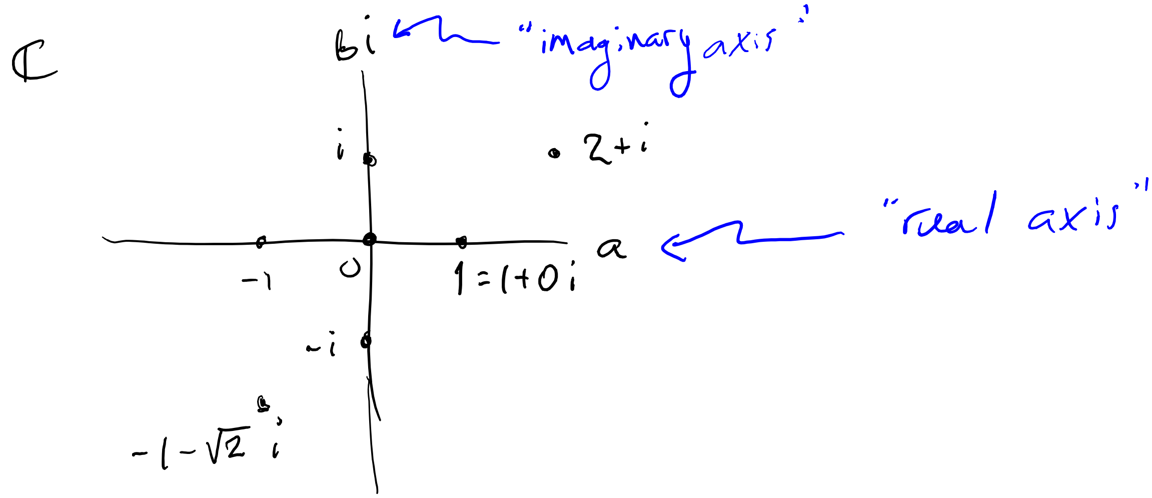
We have $(a+bi) + (c+di) = (a+b) + (c+d)i$

$$(a+bi)\cdot(c+di) = (ac - bd) + (ad+bc)i$$

$$= ac + a \cdot di + bci + bd(i)^2$$

$$= (ac - bd) + (ad + bc)$$

Note $(a+bi)^{-1} = \dfrac{a - bi}{a^2+b^2} = \dfrac{a}{a^2+b^2} - \dfrac{b}{a^2+b^2} i$.



Defn $\mathrm{Re} : \mathbb{C} \longrightarrow \mathbb{R}$, $\mathrm{Re}(a+bi) = a$
is the <u>real part</u> of $a+bi$

$\mathrm{Im} : \mathbb{C} \longrightarrow \mathbb{R}$, $\mathrm{Im}(a+bi) = b$
is the <u>imaginary part</u> of $a+bi$.

$\overline{(\ )} : \mathbb{C} \longrightarrow \mathbb{C}$

$\overline{a+bi} = a - bi$ is the <u>complex conjugate</u> of $a+bi$.

Thm If $x = a+bi$, then $x \cdot \bar{x} = a^2 + b^2$
and $x^{-1} = \dfrac{\bar{x}}{x \cdot \bar{x}} = \dfrac{a-bi}{a^2+b^2}$

Also, $\forall x, y \in \mathbb{C}$:

1. $\overline{(\bar{x})} = x$
2. $\overline{(x+y)} = \bar{x} + \bar{y}$
3. $\overline{x \cdot y} = \bar{x} \cdot \bar{y}$
4. $x = \bar{x} \iff x \in \mathbb{R}$

$$\left[\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{C} \\ a & \longmapsto & a + 0 \cdot i \end{array}\right]$$

5. $x = 0 \iff \bar{x} = 0$.