Lecture 20

Friday, February 27, 2015 8:02 AM

Thm F an ordered field, let

$F^+ = \{x \in F \mid x > 0\}$, $F^- = \{x \in F \mid x < 0\}$.

1. $x \in F^+ \iff -x \in F^-$ and $x \in F^- \iff -x \in F^+$
2. $1 \in F^+$.

Pf 1. $x \in F^+ \iff 0 < x$. Add $(-x)$ to both sides:

$$0 + (-x) < x + (-x)$$

$$-x < 0.$$

Reversing these steps, $-x < 0 \implies x > 0$.

Other statement: similar.

2. Note $1 \neq 0$. Assume for contradiction $1 \notin F^+$.

Then, by trichotomy, $1 \in F^-$.

Then $1 < 0 \overset{1}{\implies} -1 > 0$. Mult by -1:

$$(-1)(-1) > 0 \cdot (-1)$$

$$1 > 0$$

This contradicts trichotomy. $\square$

Absolute value F ordered field

Defn $|\ |: F \longrightarrow F$

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Thm "Standard" theorems about absolute value hold in an arbitrary ordered field. In particular, the triangle inequality holds:

$$|x+y| \leq |x| + |y|.$$

Defn $S \subseteq F$ then, if it exists, $\sup(S)$ is the least upper bound of S; $\inf(S)$ is the greatest lower bound of S.

e.g. $S = \{x \in \mathbb{Q} \mid 0 < x < 2\}$

$\sup(S) = 2$

$\inf(S) = 0$

$S = \{\frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\}$

$\sup(S) = 1$

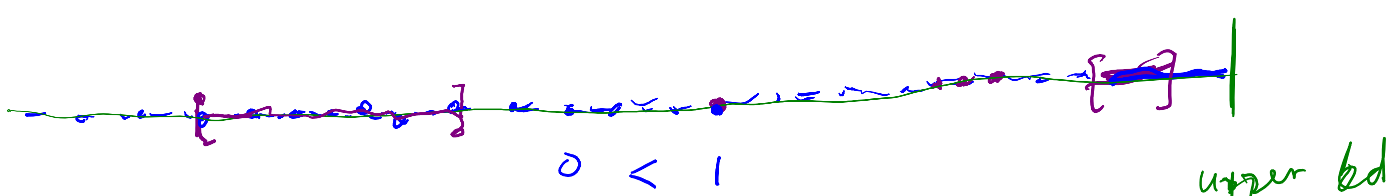
$\underbrace{\inf(S) = 0}_{\mathbb{Q} \text{ is Archimedean}}.$

__Defn__ An ordered field F is __complete__ if for every nonempty bounded above subset $S \subseteq F$, $\sup(S)$ exists as an elt of F.

__Prop__ If F is complete & $S \subseteq F$ is bdd below, then $\inf(S)$ exists as an elt of F.

__Pf Idea__ $-S = \{-s \mid s \in S\}$ is bdd above so it has a $\sup(-S) = A$. Then [check!] $-A = \inf(S)$. $\square$

What is completeness? __A__ Not having holes in the "number line" for F:



Completeness says $\sup(\sim)$ is one of the blue dots (elts of F).

Thm $\mathbb{Q}$ is **not** complete.

Note $\mathbb{R}$, the real numbers, is the smallest complete ordered field containing $\mathbb{Q}$.

Pf Suffices to find $\emptyset \neq S \subseteq \mathbb{Q}$ bdd above w/out a supremum in $\mathbb{Q}$.

Take $S = \{x \in \mathbb{Q} \mid 0 < x, \ x^2 < 2\}$,

Claim 1 $S \neq \emptyset$ and bdd above.

Claim 2 If $u = \sup(S)$, then $u^2 = 2$

Claim 3 $\forall q \in \mathbb{Q}$, $q^2 \neq 2$.

Claim 1: $1 \in S \neq \emptyset$.

We claim 3 is an upper bd of S.

To see this, note that $x, y > 0$ & $x^2 < y^2$, then $x < y$: By contrapositive: assume $x \geq y$ & $x, y > 0$. Mult by x: $x^2 \geq xy$

Mult by y: $xy \geq y^2$

Transitivity: $x^2 \geq y^2$. ✓

Now if $x \in S$ so $0 < x$ & $x^2 < 2$.

Then $2 < 3^2 = 9$ so we have $x^2 < 3^2 \Rightarrow x < 3$

i.e. 3 is an upper bound of S. ✓

Claim 3 Proof by contradiction: assume for contradiction $\exists q \in \mathbb{Q}$ s.t. $q^2 = 2$.

Write $q = \frac{a}{b}$ in least terms (a & b share no common factors). Know $\frac{a^2}{b^2} = 2$ so $a^2 = 2b^2$,

i.e. a^2 is even. Thus a is even (b/c $\text{odd}^2 = \text{odd}$). So we can write $a = 2l$, for some integer l.

$$(2l)^2 = 2b^2$$

$$4l^2 = 2b^2$$

$$2l^2 = b^2$$

i.e. b^2 is even so b is even.

This contradicts $\frac{a}{b}$ in least terms (a, b share a factor of 2!). $\square$

Thm There exists a unique complete ordered field $\mathbb{R}$.

Any two complete ordered fields F, F' are in fact "ordered isomorphic".

I.e. $\exists f : F \longrightarrow F'$

s.t. $f(x+y) = f(x) + f(y)$

$f(x \cdot y) = f(x) \cdot f(y)$

if $x < y$, then $f(x) < f(y)$

and f is bijective.

Back to Claim 2: If $S = \{x \in \mathbb{Q} \mid 0 < x,\ x^2 < 2\}$ and $u = \sup(S)$, then $u^2 = 2$. Proving this will complete our proof that $\mathbb{Q}$ is not complete!

Lemma ① If $0 < x \in \mathbb{Q}$ & $x^2 < 2$, then $\exists y \in \mathbb{Q}$ s.t. $x < y$ & $y^2 < 2$.

② If $0 < x \in \mathbb{Q}$ & $x^2 > 2$, then $\exists y \in \mathbb{Q}$ s.t. $x > y$ & $y^2 > 2$.

Note that the lemma will imply <u>Claim 2</u> : If $U^2<2$, then $\exists y\in\mathbb{Q}$, $y^2<2$, $y>U$, contradicting (by ①) U an upper bd of S. Thus $U^2\geq 2$. Similarly, ② implies $U^2\leq 2$, so by trichotomy, $U^2=2$.

It remains to prove the lemma:

<u>Pf Lemma</u> For ①, assume $0<x\in\mathbb{Q}$, $x^2<2$. Then

$$2x+1>0 \ \& \ 2-x^2>0 \text{ so } \frac{2x+1}{2-x^2}>0.$$

Let N be an integer s.t. $N>\frac{2x+1}{2-x^2}$ (use Archimedean property of $\mathbb{Q}$). Then $\frac{N}{2x+1}>\frac{1}{2-x^2}$

$$\Rightarrow \frac{2x+1}{N}<2-x^2$$

$$\Rightarrow \frac{2}{N}x+\frac{1}{N}<2-x^2$$

$$\Rightarrow x^2+\frac{2}{N}x+\frac{1}{N}<2$$

Since $\frac{1}{N^2}=\left(\frac{1}{N}\right)^2<\frac{1}{N}$, get $\left(x+\frac{1}{N}\right)^2=x^2+\frac{2}{N}x+\frac{1}{N^2}<$ $x^2+\frac{2}{N}x+\frac{1}{N}<2$, i.e. $\left(x+\frac{1}{N}\right)^2<2$.

Thus $y=x+\frac{1}{N}\in\mathbb{Q}$, $x<y$, $y^2<2$, as desired.

The proof of ② is similar. $\blacksquare$ $\mathbb{Q}$ is not complete! $\square$