Lecture 14

Tuesday, February 17, 2015 8:01 AM

Goals

- What is a number?
- Prove that $1+1=2$.

- An abstract "value" or "quality" representing "how many"
- Provide quantitative measurements
- The measurements have "units" — things in the same category — but numbers don't: capture quantity w/o reference to category.

Inductive sets

$\emptyset$	$\{\emptyset\}$	$\{\emptyset, \{\emptyset\}\}$	$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
0	1	2	3
	$0 \cup \{0\}$	$1 \cup \{1\}$	$2 \cup \{2\}$

<u>Def'n</u> For any set S, the <u>successor</u> of S is

$$S^+ = S \cup \{S\}.$$

<u>Note</u> Self-referentially builds in "1 more" element.

$0 = \emptyset$, $1 = 0^+$, $2 = 1^+$, $3 = 2^+$, ...

<u>Def'n</u> A set J is <u>inductive</u> if:

① $\emptyset \in J$

② For all $n \in J$, $n^+ \in J$.

<u>Axiom</u> There exists an inductive set.

<u>Thm</u> Suppose S is a set containing inductive subsets. Then the intersection of these subsets is also inductive.

<u>Pf</u> Any ind. subset contains $\emptyset$ by ①, thus their intersection contains $\emptyset$ as well. If n is in the intersection, then n belongs to all the ind. subsets. By ②, $n^+ \in$ all ind. subsets $\Longrightarrow n^+ \in$ intersection of ind. subsets. $\square$

<u>Defn</u> Let J be some inductive set. Let $\mathbb{N}$ be the intersection of all inductive subsets of J. This is the set of <u>natural numbers</u>.

<u>Rmk</u> $\mathbb{N}$ is the <u>smallest</u> (under $\subseteq$) ind. subset of J.

<u>Rmk</u> The defn depend on J — $\mathbb{N}$ nonetheless is independent of J.

<u>Thm</u> 0 is <u>not</u> the successor of any elt of $\mathbb{N}$. Every $n \in \mathbb{N} \setminus \{0\}$ <u>is</u> the successor of some elt of $\mathbb{N}$.

<u>Pf</u> $0 = \emptyset$ is clearly not the successor $S^+ = S \cup \{S\}$ of any S. Now assume for contradiction that $n \in \mathbb{N} \setminus \{0\}$ is not the successor of any elt of $\mathbb{N}$. Then $\mathbb{N} \setminus \{n\}$ is an inductive subset of $\mathbb{N}$. This is a contradiction by minimality of $\mathbb{N}$ amongst ind subsets. $\square$

Induction Theorem Suppose P is a property depending on (some) elements of $\mathbb{N}$. Suppose

① $P(0)$ is true

② $\forall n \in \mathbb{N}, [P(n) \Longrightarrow P(n^+)]$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Pf Let $T = \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$. Suffices to show T is an inductive subset of $\mathbb{N}$, whence $T = \mathbb{N}$ & we're done. $0 \in T$ since $P(0)$ is true. Now suppose $n \in T$ so that $P(n)$ ~~$= T$~~ is true. Then $P(n^+)$ ~~$= T$~~ is true by ②. Thus $n^+ \in T$, proving T is inductive. $\square$

Thm $\forall n \in \mathbb{N}, 0 \in n^+$.

Pf Set $T = \{n \in \mathbb{N} \mid 0 \in n^+\}$. $0^+ = 0 \cup \{0\}$ $= \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\} \ni 0$ thus $0 \in T$.

Assume $n \in T$ so that $0 \in n^+$. Is $n^+ \in T$?

$0 \in n^+ \subseteq n^+ \cup \{n^+\} = (n^+)^+$ thus $0 \in (n^+)^+$ so $n^+ \in T$. Thus T inductive so $T = \mathbb{N}$. $\square$