MATH 112: EXAM 2 REVIEW EXERCISES

Warning: These review exercises are not comprehensive. They attempt to hit some highlights from weeks 7–11 of the course. Some of the exercises are harder than exam problems. Some of them are easier. *One* of them will be an exam problem. These exercises will not be collected or graded, but you are encouraged to discuss your solutions with your peers and the instructor.

Exercise 1. For $r \geqslant 0$ and $0 \leqslant \theta < 2\pi$, let $z(r, \theta)$ denote the complex number with absolute value r which is θ radians counterclockwise from the positive real axis. Let $A = \{z(r, \theta) \mid 1 \leqslant r < 2, 0 \leqslant \theta \leqslant \pi/2\}$.

(a) Is A open? closed? neither? Explain why.
(b) What is the boundary of A?
(c) What is the interior of A?
(d) Draw a picture of A. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^3 - i$. Draw a picture of $f(A)$.

Exercise 2. For each of the following sequences, either guess the limit of the sequence or guess that the sequence diverges. Use an ε-N proof to show that your guess is correct. (In these problems, *do not* use any of the limit theorems.)

(a)
$$\lim_{n\to\infty} \frac{2n}{n+1}$$
(b)
$$\lim_{n\to\infty} \frac{5\cos(n)}{n}$$
(c)
$$\lim_{n\to\infty} \frac{n}{i^n}$$

Exercise 3. Use limit theorems to evaluate
$$\lim_{n\to\infty} \frac{5n^{-2}}{-7n^{-2} + n^{-1}}.$$
Show your work.

Exercise 4. Let $f : [a, b] \to [a, b]$ be a continuous function. Prove that there exists $c \in [a, b]$ such that $f(c) = c$. (Such a c is called a *fixed point* of f.)

Exercise 5. For each positive integer k, let x_k be an integer between 0 and 9.

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(a) Prove that
$$\sum_{k=1}^{\infty} \frac{x_k}{10^k}$$
converges.
(b) What is the relationship between the above series and decimals?
(c) Fund the sum if $(x_k)_{k=1}^{\infty} = (1, 3, 5, 1, 3, 5, 1, 3, 5, \ldots)$.
(d) Define what it means for x_k to be eventually periodic. Prove that $\sum_{k=1}^{\infty} \frac{x^k}{10^k}$ is a rational number whenever x_k is eventually periodic.

Exercise 6. Assume that
$$\sum_{k=1}^{\infty} a_k$$
converges. Prove that
$$\sum_{k=1}^{\infty} \frac{a_k}{k}$$
converges.

Exercise 7. Determine whether each of the following series converges or diverges. Prove your answer.

(a)
$$\sum_{k=1}^{\infty} \frac{k^2+1}{k^4+1}$$
(b)
$$\sum_{k=1}^{\infty} \frac{5^k+1}{k!}$$
(c)
$$\sum_{k=1}^{\infty} (\arctan(k))^k$$
(d)
$$\sum_{k=1}^{\infty} \frac{(\arctan(k))^k}{2^k}$$