MATH 112: EXAM 1 REVIEW EXERCISES

Warning: These review exercises are not comprehensive! They attempt to hit some highlights from the first six weeks of the course. Some of the exercises are harder than exam problems. Some of them are easier. *Parts of two* of them *are* exam problems. These exercises will not be collected or graded, but you are encouraged to discuss your solutions with your peers and the instructor.

Exercise 1. Let F be an ordered field. Consider the statement "Every positive element of F has a square root."

(a) Rewrite the statement using formal logical symbols and quantifiers ($\forall$, $\exists$, *etc*.).
(b) Write the negation of the statement using formal logical symbols and quantifiers.
(c) Rewrite your statement in (b) as a natural language statement.
(d) Suppose that $F = \mathbb{Q}$, the ordered field of rational numbers. Is the original statement true or false? If false, provide a counterexample.
(e) Suppose that $F = \mathbb{R}$, the ordered field of real numbers. Is the original statement true or false? If false, provide a counterexample.

Exercise 2. Use mathematical induction and Pascal's identity to prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \tag{1}$$

for $n \geqslant 1$. (Note that the statement is *false* for $n = 0$.) Now use the identity

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

to produce a second proof of the above identity. (*Hint*: plug in a particular value of x.) Now use our interpretation of $\binom{n}{k}$ as the number of k-element subsets of an n-element set to *interpret* equation (1). Can you come up with a direct counting argument that proves the result?

Exercise 3. Let $f : A \to B$ be a function with domain A and codomain B. Define a relation $\simeq_f$ on A so that for $x, y \in A$, $x \simeq_f y$ if and only if $f(x) = f(y)$.

(a) Prove that $\simeq_f$ is an equivalence relation.
(b) Consider the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $g(x, y) = y$. Describe the equivalences classes of $\simeq_g$ in words and in a picture.

Date: 1.III.15.

(c) Consider the function $n : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $n(z) = z/|z|$. Describe the equivalence classes of $\simeq_n$ in words and in a picture.

Exercise 4. Use the axioms for a field to prove that in any field F, if $x \in F \setminus \{0\}$, then $(-x)^{-1} = -(x^{-1})$.

Exercise 5. Prove that there is no rational number x such that $x^2 = 3$. What about rational x such that $x^2 = p$, p a prime number?

Exercise 6. For each of the following sets $S \subseteq \mathbb{Q}$, determine if S is bounded above; if S is bounded above, compute $\sup(S) \in \mathbb{R}$ and argue (perhaps informally) about whether $\sup(S) \in \mathbb{Q}$.

(a) $S = \{x \in \mathbb{Q} \mid x < 1\}$
(b) $S = \{x \in \mathbb{Q} \mid x < 1\} \cup \{13/2\}$
(c) $S = \{1 - 2^n \mid n \in \mathbb{Z}\}$
(d) $S = \{1 + 2^n \mid n \in \mathbb{Z}\}$
(e) $S = \{x \in \mathbb{Q} \mid x \text{ is less than the circumference of a circle of diameter } 1\}$.