LIMITS: ε, δ, $\to$, AND ALL THAT

A *function* is a rule[1] which assigns to each member of a set of inputs (the *domain*) a unique output (a member of another set, the *range* or *codomain*). We write $f : A \to B$ if f is a function with domain A and range B. In calculus, we will almost always consider functions $f : A \to B$ for which A and B only contain real numbers. Such functions are called *real functions of a single variable* and they are the objects of study in *calculus*. Since this is a calculus course, we will henceforth use the word *function* as a synonym for *real function of a single variable*, but you are encouraged to remember that the word has a more general meaning.

A *limit* is a mathematician's way of understanding the *local* behavior of a function. The limit as $x \to a$ (read "x approaches a") of a function f tells us how the output $f(x)$ of f behaves when x is very close to (but not equal to) a. (*Warning*: The arrow in "$x \to a$" has a different meaning from the arrow in "$f : A \to B$.") Below, Definition 2 will make this notion precise. But before we get there, we need to know some things about the absolute value function.

Absolute values. Let $\mathbb{R}$ denote the set of real numbers. The *absolute value function* $|\ | : \mathbb{R} \to \mathbb{R}$ takes the following values:

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

This is an example of a *piecewise* definition. The value of $|x|$ is expressed in one way when x is nonnegative (in this case, $|x| = x$), and in another way when x is negative (then $|x| = -x$). Take some time to graph this function if you don't immediately know what it looks like.

In Definition 2 we will encounter expressions of the form $|w - z|$ and it will behoove us to interpret such expressions in a particular fashion. First, let's go back to $|x|$. If we think of x as a point on the real number line, then $|x|$ measures the distance of x from the origin, 0. Now think about $|z - w|$ (where z and w are both real numbers). If $z = w$, then $|z - w| = 0$. As z moves away from w, $|z - w|$ takes positive values. In fact, $|z - w|$ measures the distance of z from w, just as $|x| = |x - 0|$ measures the distance of x from 0. (If this feels unfamiliar, draw a graph of $|x - 5|$ in order to convince yourself that this function measures the distance of x from 5 on

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[1] Please refrain from taking the term *rule* too seriously. Potential synonyms include *assignment* and *list*. Potential philosophical debates are many.

the real number line. In these terms, what does $|x+5|$ measure? [Hint: $x+5 = x-(-5)$.])

When manipulating absolute values, we will frequently invoke four important properties.

Theorem 1. *For any real numbers x, y, a and any $R > 0$ we have*

(0) $|x| = 0$ *if and only if* $x = 0$,
(R) $|x-a| < R$ *if and only if* $a - R < x < a + R$,
(M) $|x \cdot y| = |x| \cdot |y|$,
(Δ) $|x+y| \le |x| + |y|$.

Property (M) is known as *multiplicativity*, and (Δ) is called the *triangle inequality* or, alternatively, *subadditivity*. You'll be asked to justify these properties in the exercises, but take some time to make sure you believe them right now. Properties (M) and (Δ) might seem pretty obvious if x and y are both positive, but what about if they're both negative? What if the signs are mixed? Is Theorem 1 more or less obvious when you think of it in terms of distance, or in terms of the piecewise definition of the absolute value?

We are now ready to define what a limit is.

Definition 2. We say that L is the *limit of f as $x \to a$* and write

$$L = \lim_{x \to a} f(x) \quad \text{or} \quad L = \lim_{x\to a} f(x)$$

or

$$f(x) \to L \text{ as } x \to a$$

if and only if the following condition holds:

> For every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Let's spend a few moments unpacking this definition.

If and only if. The phrase *if and only if* indicates logical equivalence. Whenever we write $L = \lim_{x\to a} f(x)$, we mean that the condition holds, and similarly whenever the condition holds, it is true that $L = \lim_{x\to a} f(x)$.

For every real number $\varepsilon > 0$. The phrase *for every* represents the "universal quantifier." The symbol ε is the Greek letter *epsilon*. No matter which positive real number ε we are given, we're supposed to be able to make the rest of the statement true.

There exists a real number $\delta > 0$. The phrase *there exists* represents the "existential quantifier." The symbol δ is the Greek letter *delta*. Now the game is on: no matter which positive ε we're given, some positive δ is supposed to exist such that the subsequent statement is true.

Think of "for every…there exists…" statements as a game with two teams: the universal team, and the existential team. The universal team (consisting of very smart and very incredulous individuals) is going to produce some (any!) $\varepsilon > 0$. An even smarter team (of which you are a member) is going to then produce a $\delta > 0$ that will work *despite* the cleverness of the universal team.

Such that. The phrase *such that* indicates a transition in our mathematical sentence. The quantifiers are now established and what follows is the statement that must be satisfied.

If $0 < |x - a| < \delta$**.** This statement tells us we are assuming that "$0 < |x - a| < \delta$" is a true statement. But what does $0 < |x - a| < \delta$ mean? It's a compound mathematical phrase telling us that both $0 < |x - a|$ and $|x - a| < \delta$ are true. Recall our comments on absolute values before: this is a statement about the distance of x from a. Namely, $|x - a| < \delta$ tells us that x is less than distance δ from a, and $0 < |x - a|$ tells us that x is some positive distance from a, *i.e.*, $x \neq a$.

Then $|f(x) - L| < \varepsilon$**.** This is now the conclusion of our logical / mathematical sentence. The phrase $|f(x) - L| < \varepsilon$ must hold whenever our "if" clause (namely, $0 < |x - a| < \delta$) holds.

This concludes our unpacking, but what does it all mean? It's not too bad: first, we're given a target, an arbitrary $\varepsilon > 0$. There's then supposed to be some $\delta > 0$ such that whenever x is at most distance δ from a (but not equal to a), we know that $f(x)$ is at most ε from L.

Spend some time meditating on Definition 2 and the above explanation and convince yourself that the definition is reasonable. Try drawing a picture to make sense of things. If it's all a little mysterious, think on the following: if ε is really big, it shouldn't be hard to make $f(x)$ ε-close to L. But if ε is super small (say $\varepsilon = 0.000000001$, or smaller) it gets a lot harder to make $f(x)$ within ε of L. Thus, despite the fact that the definition talks about *any* $\varepsilon > 0$, the real action happens when ε is a very small positive number. (Our proof of Proposition 5 below will put a finer point on this observation.)

ε-δ **proofs.** Now that we've accommodated ourselves to the definition of a limit, let's see it in action. For our first example, let's let f be the real function of a single variable which assigns the value $5x-2$ to the real number x. We can write $f : \mathbb{R} \to \mathbb{R}$ (where $\mathbb{R}$ is notation for the set of all real numbers), and $f(x) = 5x - 2$ or $f : x \mapsto 5x - 2$ to succinctly capture this information.

Proposition 3. *The limit of f as x approaches* 1 *is* 3, i.e.,

$$\lim_{x\to 1} (5x - 2) = 3.$$

Before we launch into a proper proof, let's first ask ourselves if Proposition 3 makes sense. Does it? Well, sure. If you plug $x = 0.9, 0.99, 0.999,$ *&c* into f, you get numbers successively closer and closer to 3. The same thing happens if you approach 1 from the other side, *e.g.*, trying $x = 1.1$, 1.01, 1.001, *&c*. Not so coincidentally, we also have $f(1) = 5 \cdot 1 - 2 = 3$. This is an example of *continuity*, a subject we will study in great depth as we proceed through the course.

It's also possible to see that $5x - 2 \to 3$ as $x \to 1$ graphically. Try drawing a graph to see why this makes sense.

In order to successfully write an ε-δ proof of Proposition 3, we probably want to start with some thinking and scratch work.

Scratch work. The universal team hands us an $\varepsilon > 0$. We seek $\delta > 0$ such that whenever $0 < |x - 1| < \delta$, it also happens that $|f(x) - 3| < \varepsilon$. Let's expand $|f(x) - 3|$ to see what we're really after:

$$\begin{aligned} |f(x) - 3| &= |(5x - 2) - 3| \\ &= |5x - 5| \\ &= 5|x - 1| \end{aligned}$$

Aha! We've managed to massage $|f(x) - 3|$ until a nice, pleasant-looking $|x - 1|$ appeared. This is a good thing because $|x - 1|$ is precisely what we're able to control with our choice of δ.

Remember that we're trying to find $\delta > 0$ such that $0 < |x - 1| < \delta$ implies that $|f(x) - 3| = 5|x - 1| < \varepsilon$. Well, $5|x - 1| < \varepsilon$ is equivalent to $|x - 1| < \varepsilon/5$, so can can choose $\delta = \varepsilon/5$. Indeed, if $|x - 1| < \varepsilon/5$, then

$$|f(x) - 3| = 5|x - 1| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon,$$

as desired. ♯

Our scratch work is a little sloppy, so we should rewrite it in our final proof in order to deliver a succinct and convincing argument that Proposition 3 is true.

Proof. Suppose $\varepsilon > 0$.[2] Set $\delta = \varepsilon/5$ and assume $0 < |x - 1| < \delta = \varepsilon/5$. Then

$$\begin{aligned} |f(x) - 3| &= |(5x - 2) - 3| \\ &= |5x - 5| \\ &= 5|x - 1| \\ &< 5 \cdot \frac{\varepsilon}{5} \\ &= \varepsilon. \end{aligned}$$

By the definition of a limit, we have verified that

$$\lim_{x \to 1} f(x) = 3.$$

□

Let's do it again, but this time with an even easier function.

[2]This is slightly formalized language that signals to the reader that we are taking an arbitrary $\varepsilon > 0$ — whichever one the universal team decided to hand us.

Proposition 4. *Suppose $c : \mathbb{R} \to \mathbb{R}$ is the function which takes x to a constant number a for every real number x,* i.e., *$c(x) = a$. Suppose b is any real number. Then*

$$\lim_{x \to b} c(x) = a.$$

Scratch work. Given an arbitrary $\varepsilon > 0$, we want to find δ such that $0 < |x - b| < \delta$ implies that $|c(x) - a| < \varepsilon$. Let's expand $|c(x) - a|$:

$$|c(x) - a| = |a - a| = 0.$$

Whoa! That was easy. We always have $0 < \varepsilon$, so we can take any positive δ! $\sharp$

Proof. Suppose $\varepsilon > 0$. Then $|c(x) - a| = |a - a| = 0 < \varepsilon$, so

$$\lim_{x \to b} a = a$$

by the definition of a limit. $\square$

Note that we could have chosen δ to be 1 or π or 10^8, but such a choice was immaterial to the proof.

Also note that we had to be careful with notation in Proposition 4 and its proof. Looking back at Definition 2, we had to carefully substitute $c(x)$ for $f(x)$, a for L, and b for a. While this can be slightly annoying, it shouldn't bother us: *mathematics is invariant under choice of notation*.

Perhaps Proposition 4 was too easy and we should sink our teeth into a slightly harder problem.

Proposition 5. *Let $g : \mathbb{R} \to \mathbb{R}$ be the function which assigns x to $g(x) = x^2$. Then*

$$\lim_{x \to -3} g(x) = 9.$$

Scratch work. Given $\varepsilon > 0$ we want to find $\delta > 0$ such that $0 < |x - (-3)| < \delta$ implies that $|g(x) - 9| < \varepsilon$. Well,

$$\begin{aligned} |g(x) - 9| &= |x^2 - 9| \\ &= |(x+3)(x-3)| \\ &= |x+3||x-3|. \end{aligned}$$

We want $|x - (-3)| = |x + 3|$ small to imply that $|x + 3||x - 3|$ is small. That should be fine unless $|x - 3|$ can get huge when $|x + 3|$ becomes small. But when $|x + 3|$ is small, x is close to -3, so $|x + 3|$ is close to 6. Since $0 \cdot 6 = 0$, we should be fine, but we need to formalize this intuition.

Let's start by asserting that $0 < \delta \leq 1$. (We can do this because we're on the existential team and we get to choose δ; let's choose it so it's always positive and at most 1.) If $|x + 3| < \delta \leq 1$, then $-4 \leq x \leq -2$. Subtracting 3 from this system of inequalities, we find that $-7 \leq x - 3 \leq -5$, whence

$|x-3| \leq 7$. (Pause and make sure these manipulations make sense, recalling Theorem 1(R).) So as long as $|x+3| < \delta \leq 1$, we know that

$$|g(x) - 9| = |x+3||x-3| < \delta \cdot 7.$$

Thus if we further guarantee that $\delta \leq \varepsilon/7$, we'll be done! Can we do this? Sure: just set δ to be the minimum of 1 and $\varepsilon/7$; then both $\delta \leq 1$ and $\delta \leq \varepsilon/7$ are true statements. $\sharp$

In our formal proof, we'll use the following notation. If z and w are real numbers, define

$$\min\{z, w\} = \begin{cases} z & \text{if } z \leq w; \\ w & \text{if } w \leq z \end{cases}$$

to be the *minimum* of z and w.

Proof of Proposition 5. Suppose $\varepsilon > 0$. Set $\delta = \min\{1, \varepsilon/7\}$ and assume that $0 < |x - (-3)| < \delta$. Thus $|x+3| < 1$ and $|x+3| < \varepsilon/7$. It follows that

$$\begin{aligned} |g(x) - 9| &= |x^2 - 9| \\ &= |x+3||x-3| \\ &< \frac{\varepsilon}{7} \cdot 7 \\ &= \varepsilon. \end{aligned}$$

The inequality is justified because $|x+3| < 1$ implies that $-4 < x < -2$, whence $-7 < x - 3 < -5$ and $|x-3| < 7$. Thus by the definition of a limit, $g(x) \to 9$ as $x \to -3$. $\square$

Exercises. These exercises are due on Friday, 6.IX.14 at the start of class. Please read the homework policy section of the syllabus before embarking on this assignment.

Exercise 1. Prove [3] Theorem 1.

Exercise 2. Let $h : \mathbb{R} \to \mathbb{R}$ denote the function taking $x \mapsto -2x + 3$.

(a) Draw a graph of $y = h(x)$ with x ranging from 0 to 2. (Make sure your graph is large and legible.) Suppose $\varepsilon = 1/2$ and draw a horizontal band that contains the y-values within ε of 1. Use your picture to estimate a δ such that $0 < |x-1| < \delta$ implies that $|h(x) - 1| < \varepsilon = 1/2$. Draw a vertical band consisting of x-values

[3] A *proof* is a sequence of English and mathematical sentences logically justifying a particular statement. In particular, a proof is something that another person with similar training can *read*, not just decode. You may have to put serious thought and scratch work into your proof of Theorem 1, but your completed proof should be a record of logically necessary steps leading to the conclusion of the theorem; moreover, your proof should *explain* your logic. For each of the four statements in Theorem 1, the most likely starting place for your proof will be the *definition* of the absolute value function.

N.B. Writing proofs is hard! Please talk to me, a peer, a tutor, or a more experienced math student if you have questions.

satisfying $0 < |x - 1| < \delta$ and explain (in one or two complete sentences) how your picture justifies your choice of δ.

(b) Guess the limit of $h(x)$ as $x \to 1$ and make a table of values of $h(x)$ for x close to 1. (Make sure to include x both less than and greater than 1.) Write a sentence explaining how your table of values supports your guess.

(c) Write an ε-δ proof verifying your guess for $\lim_{x\to 1} h(x)$ from part (b). *Do not* include your scratch work.

Exercise 3. Not every function has a limit as $x \to a$ for every a. For each of the following functions, determine the set of a such that the limit of the function as $x \to a$ does not exist. Draw a picture and explain how it justifies your claim. Also write a few sentences that use the ε-δ definition of a limit to justify your claim.

(a)
$$f(x) = \begin{cases} 0 & \text{if } x \le 1; \\ 1 & \text{if } x > 1. \end{cases}$$

(b)
$$g(x) = \frac{3}{x+2}.$$

(c)
$$h(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for } n \text{ some positive integer}; \\ 0 & \text{otherwise.} \end{cases}$$