MATH 111: INTEGRALS

Let f be a bounded function on a closed interval $[a, b]$.

Definition 1. A *partition* of $[a, b]$ is a set $\mathcal{P} = \{t_0, t_1, \dots, t_n\}$ such that

$$a = t_0 < t_1 < \cdots < t_n = b.$$

Definition 2. Let $\mathcal{P} = \{t_0, t_1, \dots, t_n\}$ be a partition of $[a, b]$. For each $i = 1, 2, \dots, n$ let

$$m_i = \inf f([t_{i-1}, t_i]),$$
$$M_i = \sup f([t_{i-1}, t_i]).$$

Then the *lower sum* of f relative to $\mathcal{P}$ is

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

and the *upper sum* of f relative to $\mathcal{P}$ is

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Definition 3. Define the numbers

$$L_a^b(f) = \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\},$$
$$U_a^b(f) = \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\}.$$

We say that f is *integrable* if $L_a^b(f) = U_a^b(f)$, in which case this common value is called the *integral* of f from a to b; it is denoted

$$\int_a^b f.$$

We can now build up a body of propositions, lemmas, and theorems surrounding the notions of integrability and integrals.

Proposition 4. *For any partition $\mathcal{P} = \{t_0, \dots, t_n\}$ of $[a, b]$,*

$$L(f, \mathcal{P}) \le U(f, \mathcal{P}).$$

Proof. Since $m_i \le M_i$ for all i, we have that $m_i(t_i - t_{i-1}) \le M_i(t_i - t_{i-1})$ for all i. Thus

$$L(f, \mathcal{P}) = \sum m_i(t_i - t_{i-1}) \le \sum M_i(t_i - t_{i-1}) = U(f, \mathcal{P}).$$

□

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We now aim to compare lower and upper sums for different partitions. In order to make these comparisons, we will need the notion of a refinement of a partition.

Definition 5. Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of $[a, b]$. If $\mathcal{P} \subseteq \mathcal{P}'$, then we call $\mathcal{P}'$ a *refinement* of $\mathcal{P}$.

Proposition 6. *Let $\mathcal{P}$, $\mathcal{P}'$ be partitions of $[a, b]$. If $\mathcal{P}'$ refines $\mathcal{P}$, then*

$$L(f, \mathcal{P}) \leq L(f, \mathcal{P}') \leq U(f, \mathcal{P}') \leq U(f, \mathcal{P}).$$

Proof. Manifest if you draw a picture. □

Proposition 7. *If $\mathcal{P}$ and $\mathcal{Q}$ are any two partitions, then*

$$L(f, \mathcal{P}) \leq U(f, \mathcal{Q}).$$

Proof. Let $\mathcal{P}' = \mathcal{P} \cup \mathcal{Q} = \{t \mid t \in \mathcal{P} \text{ or } t \in \mathcal{Q}\}$. Then $\mathcal{P}'$ refines $\mathcal{P}$ and $\mathcal{Q}$. Thus (using both Proposition 6 and Proposition 4)

$$L(f, \mathcal{P}) \leq L(f, \mathcal{P}') \leq U(f, \mathcal{P}') \leq U(f, \mathcal{Q}).$$

□

Corollary 8. *We always have $L_a^b(f) \leq U_a^b(f)$.*

Proof. Let $\mathcal{Q}$ be a partition of $[a, b]$. If $\mathcal{P}$ is any other partition, Proposition 7 tells us that $L(f, \mathcal{P}) \leq U(f, \mathcal{Q})$. Thus $U(f, \mathcal{Q})$ is an upper bound for the set of all lower sums. It follows that

$$L_a^b(f) = \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\} \leq U(f, \mathcal{Q}).$$

In turn, since $\mathcal{Q}$ was arbitrary, this inequality says that $L_a^b(f)$ is a lower bound for the set of all upper sums. Hence

$$L_a^b(f) \leq \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\} = U_a^b(f),$$

as desired. □

Having built up some useful ways for comparing lower and upper sums, we now turn to the task of proving that all continuous functions are integrable. We will need a lemma and a proposition to get the ball rolling.

Lemma 9. *Let $X \subseteq \mathbb{R}$.*

(1) If $\sup X$ exists, then for any $\varepsilon > 0$, there exists $x \in X$ such that

$$0 \leq \sup X - x < \varepsilon.$$

(2) If $\inf X$ exists, then for any $\varepsilon > 0$, there exists $x \in X$ such that

$$0 \leq x - \inf X < \varepsilon.$$

Proof. Given $\varepsilon > 0$, observe that $\sup X - \varepsilon$ is *not* an upper bound for X. (Otherwise, $\sup X$ would not be the *least* upper bound of X.) Thus there exists $x \in X$ such that $\sup X - \varepsilon < x$, whence $\sup X - x < \varepsilon$. We also have $0 \leq \sup X - x$ since $x \in X$ and $\sup X$ is an upper bound for X.

The proof for part (2) is similar. □

Recall that throughout this note, f is a bounded function on $[a, b]$.

Proposition 10. *The function f is integrable if and only if for all $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[a, b]$ such that*

$$0 \leq U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon.$$

Proof. First assume that f is integrable. For any partition $\mathcal{P}$, the inequality $0 \leq U(f, \mathcal{P}) - L(f, \mathcal{P})$ is guaranteed by Proposition 4. Given $\varepsilon > 0$, Lemma 9 implies that there is an element of $\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\}$ within ε of $L_a^b(f) = \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\}$. In particular, there is a partition $\mathcal{P}_1$ such that

$$L_a^b(f) - L(f, \mathcal{P}_1) < \varepsilon/2.$$

Similarly, there is a partition $\mathcal{P}_2$ such that

$$U(f, \mathcal{P}_2) - U_a^b(f) < \varepsilon/2.$$

It follows that

$$U(f, \mathcal{P}_2) - L(f, \mathcal{P}_1) < \varepsilon.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Since $\mathcal{P}$ refines both $\mathcal{P}_1$ and $\mathcal{P}_2$, Proposition 7 implies that

$$L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}) \leq U(f, \mathcal{P}) \leq U(f, \mathcal{P}_2).$$

Thus we also have

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon,$$

as desired.

We now suppose that for any $\varepsilon > 0$ there exists a partition $\mathcal{P}$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$. In order to prove that f is integrable, we must show that $L_a^b(f) = U_a^b(f)$. Given $\varepsilon > 0$, choose $\mathcal{P}$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$. Then

$$L(f, \mathcal{P}) \leq L_a^b(f) \leq U_a^b(f) \leq U(f, \mathcal{P}),$$

so

$$0 \leq U_a^b(f) - L_a^b(f) < \varepsilon$$

for all $\varepsilon > 0$. This is only possible if $L_a^b(f) = U_a^b(f)$, *i.e.*, if f is integrable. $\square$

We are just about ready to prove our first major theorem on integrability, namely that all continuous functions on a closed interval are integrable, but we will need the following definition and theorem in order to continue. For the time being, we drop the assumption that f is bounded on $[a, b]$.

Definition 11. A function f defined on a closed interval $[a, b]$ is *uniformly continuous* on $[a, b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| < \varepsilon.$$

Theorem 12. *A function f is continuous on $[a, b]$ if and only if it is uniformly continuous on $[a, b]$.*

It is obvious from the definitions that uniform continuity implies continuity. We will not undertake a proof of the opposite implication here, but — briefly engaging in a small amount of cheating — we will freely use it. The reader is encouraged to think about why such a result should be expected, and she is referred to Math 112 if she would like to see a proof.

Theorem 13. *If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.*

Proof. First note that the extreme value theorem implies that f is bounded, so we are free to invoke all of the results proved above.

Given $\varepsilon > 0$, Theorem 12 implies that there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$. Now pick any partition $\mathcal{P} = \{t_0, \ldots, t_n\}$ such that each subinterval of $\mathcal{P}$ has length less than δ. It follows that whenever $x, y \in [t_{i-1}, t_i]$, then $|x - y| < \delta$, so $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$. Thus $M_i - m_i \leq \frac{\varepsilon}{2(b-a)}$, and the following chain of (in)equalities is valid:

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \sum M_i(t_i - t_{i-1}) - \sum m_i(t_i - t_{i-1}) \\ &= \sum \left(M_i(t_i - t_{i-1}) - m_i(t_i - t_{i-1})\right) \\ &= \sum (M_i - m_i)(t_i - t_{i-1}) \\ &\leq \sum \frac{\varepsilon}{2(b-a)}(t_i - t_{i-1}) \\ &= \frac{\varepsilon}{2(b-a)} \sum (t_i - t_{i-1}) \\ &= \frac{\varepsilon}{2(b-a)}(t_n - t_0) \\ &= \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

By Proposition 10, we may conclude that f is integrable on $[a, b]$. □

Theorem 14 (Fundamental Theorem of Calculus)**.** *Suppose f is integrable on $[a, b]$ and there exists g such that $f = g'$. Then*

$$\int_a^b f = g(b) - g(a).$$

Proof. Let $\mathcal{P} = \{t_0, \ldots, t_n\}$ be any partition of $[a, b]$. Applying the mean value theorem to g over the subinterval $[t_{i-1}, t_i]$, we see that there exists $c_i \in (t_{i-1}, t_i)$ such that

$$g'(c_i) = \frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}}.$$

Since $g' = f$, we may rewrite this as

$$f(c_i)(t_i - t_{i-1}) = g(t_i) - g(t_{i-1}).$$

Since $m_i \leq f(c_i) \leq M_i$, we have that

$$L(f,\mathcal{P}) = \sum m_i(t_i - t_{i-1}) \leq \sum f(c_i)(t_i - t_{i-1}) \leq \sum M_i(t_i - t_{i-1}) = U(f,\mathcal{P}).$$

We have just seen that the middle sum can be rewritten as

$$\sum (g(t_i) - g(t_{i-1}))$$

which telescopes to give $g(b) - g(a)$. Thus for any partition $\mathcal{P}$ of $[a,b]$ we have

$$L(f,\mathcal{P}) \leq g(b) - g(a) \leq U(f,\mathcal{P}). \tag{1}$$

Since f is integrable, Proposition 10 implies that for all $\varepsilon > 0$ there exists a partition $\mathcal{P}$ such that

$$L(f,\mathcal{P}) \leq \int_a^b f \leq U(f,\mathcal{P}) \quad \text{and} \quad U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon.$$

Combining this with (1), we see that for all $\varepsilon > 0$,

$$\left| \int_a^b f - (g(b) - g(a)) \right| < \varepsilon.$$

This is only possible if

$$\int_a^b f = g(b) - g(a),$$

as desired.

□