Chapter 11

Infinite Series

11.1 Infinite Series

11.1 Definition (Series operator.) If f is a complex sequence, we define a new sequence $\sum f$ by

$$(\sum f)(n) = \sum_{j=0}^{n} f(j) \text{ for all } n \in \mathbf{N}$$

or

$$\sum\{f(n)\} = \{\sum_{j=0}^{n} f(j)\} \text{ for all } n \in \mathbf{N}.$$

We use variations, such as

$$\sum\{f(n)\}_{n\geq 1} = \{\sum_{j=1}^{n} f(j)\}_{n\geq 1}.$$

$\sum$ is actually a function that maps complex sequences to complex sequences. We call $\sum f$ the *series* corresponding to f.

11.2 Remark. If f, g are complex sequences and $c \in \mathbf{C}$, then

$$\sum(f+g) = \sum f + \sum g$$

and

$$\sum(cf) = c(\sum f),$$

since for all $n \in \mathbf{N}$,

$$\begin{aligned}\left(\sum(f+g)\right)(n) &= \sum_{j=0}^{n}(f+g)(j) = \sum_{j=0}^{n} f(j)+g(j) \\ &= \sum_{j=0}^{n} f(j) + \sum_{j=0}^{n} g(j) = (\sum f)(n) + (\sum g)(n) \\ &= (\sum f + \sum g)(n)\end{aligned}$$

and

$$\begin{aligned}\left(\sum(cf)\right)(n) &= \sum_{j=0}^{n}(cf)(j) = \sum_{j=0}^{n} c\cdot f(j) = c\sum_{j=0}^{n} f(j) \\ &= c\cdot\left((\sum f)(n)\right) = (c\cdot \sum f)(n).\end{aligned}$$

11.3 Examples. If $\{r^n\}$ is a geometric sequence, then $\sum\{r^n\} = \{\sum_{j=0}^{n} r^j\}$ is a sequence we have been calling a geometric series. If $\{c_n(t)\} = \left\{\dfrac{t^{2n}(-1)^n}{(2n)!}\right\}$, then $\sum\{c_n(t)\} = \{C_n(t)\}$ is the sequence for $\cos(t)$ that we studied in the last chapter.

11.4 Definition (Summable sequence.) A complex sequence $\{a_n\}$ is *summable* if and only if the series $\sum\{a_n\}$ is convergent. If $\{a_n\}$ is summable, we denote $\lim(\sum\{a_n\})$ by $\displaystyle\sum_{n=0}^{\infty} a_n$. We call $\displaystyle\sum_{n=0}^{\infty} a_n$ the *sum* of the series $\sum\{a_n\}$.

11.5 Example. If $r \in \mathbf{C}$ and $|r| < 1$, then $\displaystyle\sum_{n=0}^{\infty} r^n = \lim\{\sum_{j=0}^{n} r^j\} = \frac{1}{1-r}$.

11.6 Example (Harmonic series.) The series

$$\sum\left\{\frac{1}{n}\right\}_{n\geq 1} = \left\{\sum_{j=1}^{n}\frac{1}{j}\right\}_{n\geq 1}$$

is called the *harmonic series*, and is denoted by $\{H_n\}_{n\geq 1}$. Thus

$$H_n = \sum_{j=1}^{n}\frac{1}{j}.$$

We will show that $\{H_n\}_{n\geq 1}$ diverges; i.e., the sequence $\left\{\frac{1}{n}\right\}_{n\geq 1}$ is not summable. For all $n \geq 1$, we have

$$\begin{aligned}
H_{2n} &= \quad 1 \quad + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} \\
&\geq \left(\frac{1}{2} + \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \cdots + \frac{1}{2n} + \frac{1}{2n} \\
&= \frac{1}{2} + \frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \cdots + \frac{2}{2n} \\
&= \frac{1}{2} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\
&= \frac{1}{2} + H_n.
\end{aligned}$$

From the relation $H_{2n} \geq \frac{1}{2} + H_n$, we have

$$\begin{aligned}
H_2 &\geq \frac{1}{2} + H_1 = \frac{1}{2} + 1 \\
H_4 &\geq \frac{1}{2} + H_2 \geq \frac{2}{2} + 1 \\
H_8 &\geq \frac{1}{2} + H_4 \geq \frac{3}{2} + 1
\end{aligned}$$

and (by induction),

$$H_{2^n} \geq \frac{n}{2} + 1 \text{ for all } n \in \mathbf{Z}_{\geq 1}.$$

Hence, $\{H_n\}_{n\geq 1}$ is not bounded, and thus $\{H_n\}$ diverges; i.e., $\left\{\frac{1}{n}\right\}_{n\geq 1}$ is not summable.

11.7 Theorem (Sum theorem for series.) *Let f, g be summable sequences and let $c \in \mathbf{C}$. Then $f + g$ and cf are summable, and*

$$\begin{aligned}
\sum_{n=0}^{\infty} (f+g)(n) &= \sum_{n=0}^{\infty} f(n) + \sum_{n=0}^{\infty} g(n) \\
\sum_{n=0}^{\infty} cf(n) &= c\sum_{n=0}^{\infty} f(n).
\end{aligned}$$

If f is not summable, and $c \neq 0$, then cf is not summable.

Proof: The proof is left to you.

11.8 Exercise. Let f, g be summable sequences. Show that $f + g$ is summable and that

$$\sum_{j=0}^{\infty}(f+g)(j) = \sum_{j=0}^{\infty} f(j) + \sum_{j=0}^{\infty} g(j).$$

11.9 Example. The product of two summable sequences is not necessarily summable. If

$$f = \left\{1, -1, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{4}}, -\sqrt{\frac{1}{4}}, \cdots\right\}_{n\geq 1}$$

then

$$\sum f = \left\{1, 0, \sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{3}}, 0, \sqrt{\frac{1}{4}}, 0, \cdots\right\}_{n\geq 1}.$$

This is a null sequence, so f is summable and $\sum\limits_{n=1}^{\infty} f(n) = 0$. However,

$$f^2 = \left\{1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \cdots\right\}_{n\geq 1},$$

so $\left(\sum(f^2)\right)(2n) = 2\sum\limits_{j=1}^{n} \frac{1}{j} = 2H_n$. Thus $\sum(f^2)$ is unbounded and hence f^2 is not summable.

11.10 Theorem. *Every summable sequence is a null sequence. [The converse is not true. The harmonic series provides a counterexample.]*

Proof: Let f be a summable sequence. Then $\{\sum\limits_{j=0}^{n} f(j)\}$ converges to a limit L, and by the translation theorem $\{\sum\limits_{j=0}^{n+1} f(j)\} \to L$ also. Hence

$$\{\sum_{j=0}^{n+1} f(j)\} - \{\sum_{j=0}^{n} f(j)\} \to L - L = 0;$$

i.e.,

$$\{f(n+1)\} \to 0$$

and it follows that f is a null sequence. $\|$

11.2 Convergence Tests

In this section we prove a number of theorems about convergence of series of real numbers. Later we will show how to use these results to study convergence of complex sequences.

11.11 Theorem (Comparison test for series.) *Let f, g be two sequences of non-negative numbers. Suppose that there is a number $N \in \mathbf{N}$ such that*

$$f(n) \leq g(n) \text{ for all } n \in \mathbf{Z}_{\geq N}.$$

Then

if g is summable, then f is summable,

and

if f is not summable, then g is not summable.

Proof: Note that the two statements in the conclusion are equivalent, so it is sufficient to prove the first.

Suppose that g is summable. Then $\sum g$ converges, so $\sum g$ is bounded — say $(\sum g)(n) \leq B$ for all $n \in \mathbf{N}$. Then for all $n \geq N+1$,

$$\begin{aligned} \sum_{j=0}^{n} f(j) &= \sum_{j=0}^{N} f(j) + \sum_{j=N+1}^{n} f(j) \leq \sum_{j=0}^{N} f(j) + \sum_{j=N+1}^{n} g(j) \\ &\leq \sum_{j=0}^{N} f(j) + \sum_{j=0}^{n} g(j) \leq \sum_{j=0}^{N} f(j) + B. \end{aligned}$$

Since for $n \leq N$ we have

$$\sum_{j=0}^{n} f(j) \leq \sum_{j=0}^{N} f(j) \leq \sum_{j=0}^{N} f(j) + B,$$

we see that $\sum f$ is bounded by $\displaystyle\sum_{j=0}^{N} f(j) + B$. Also $\sum f$ is increasing, since $(\sum f)(n+1) = (\sum f)(n) + f(n+1) \geq \sum f(n)$. Hence $\sum f$ is bounded and increasing, and hence $\sum f$ converges; i.e., f is summable. |||

11.12 Examples. Since

$$\frac{1}{\sqrt{n}} \geq \frac{1}{n} > 0 \text{ for all } n \in \mathbf{Z}_{\geq 1}$$

and $\sum \left\{\frac{1}{n}\right\}_{n\geq 1}$ diverges, it follows that $\sum \left\{\frac{1}{\sqrt{n}}\right\}_{n\geq 1}$ also diverges. Since $\sum\{t^n\}$ converges for $0 \leq t < 1$, $\sum \left\{\frac{t^n}{n!}\right\}$ also converges for $0 \leq t < 1$.

In order to use the comparison test, we need to have some standard series to compare other series with. The next theorem will provide a large family of standard series.

11.13 Theorem. *Let $p \in \mathbf{Q}$. Then $\left\{\frac{1}{n^p}\right\}_{n\geq 1}$ is summable if $p > 1$, and is not summable if $p \leq 1$.*

Proof: Let $f_p(n) = \frac{1}{n^p}$ for $n \in \mathbf{Z}_{\geq 1}$. Then for all $n \in \mathbf{Z}_{\geq 1}$ and all $p \geq 0$,

$$\begin{aligned}
(\textstyle\sum f_p)(n) &\leq (\textstyle\sum f_p)(2n+1) \\
&= \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots + \frac{1}{(2n)^p} + \frac{1}{(2n+1)^p} \\
&\leq 1 + \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n)^p} + \frac{1}{(2n)^p} \\
&= 1 + 2\left(\frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n)^p}\right) \\
&= 1 + \frac{2}{2^p}\left(\frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p}\right) \\
&= 1 + 2^{1-p}(\textstyle\sum f_p)(n).
\end{aligned}$$

Hence,

$$(1 - 2^{1-p})\left((\textstyle\sum f_p)(n)\right) \leq 1.$$

If $p > 1$, then $1 - p < 0$, so $2^{1-p} < 1$ and $1 - 2^{1-p}$ is positive. Hence $(\sum f_p)(n) \leq \frac{1}{1-2^{1-p}}$; i.e., the sequence $\sum f_p$ is bounded. It is also increasing, so it converges.

If $p < 1$, then $\frac{1}{n^p} \geq \frac{1}{n}$, so by using the comparison test with the harmonic series, f_p is not summable. ‖

11.14 Remark. For $p > 1$, the proof of the previous theorem shows that

$$\sum_{j=1}^{\infty} \frac{1}{n^p} = \lim \sum f_p \leq \frac{1}{1 - 2^{1-p}}.$$

Hence, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{1}{1 - 2^{-1}} = 2,$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \leq \frac{1}{1 - 2^{-3}} = \frac{8}{7} = 1.1428\cdots.$$

The exact values of the series (found by Euler) are

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.6449\cdots$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.0823\cdots.$$

11.15 Examples. $\left\{\dfrac{1}{n^2 + n^{1/2}}\right\}_{\geq 1}$ is summable, since

$$0 \leq \frac{1}{n^2 + n^{1/2}} \leq \frac{1}{n^2} \text{ for all } n \in \mathbf{Z}_{\geq 1}$$

and $\left\{\dfrac{1}{n^2}\right\}$ is summable.

$\left\{\dfrac{1}{1 + n^{1/2}}\right\}_{n\geq 1}$ is not summable since

$$\frac{1}{1 + n^{1/2}} \geq \frac{1}{n^{1/2} + n^{1/2}} = \frac{1}{2} \cdot \frac{1}{n^{1/2}} \text{ for all } n \in \mathbf{Z}_{\geq 1}$$

and $\left\{\dfrac{1}{n^{1/2}}\right\}_{n\geq 1}$ is not summable.

11.16 Example. Let $w = \dfrac{3}{5} + \dfrac{4}{5}i$, and let $z \in D(0,1)$. Then $\left\{\dfrac{1}{n^2|z - w^n|}\right\}_{\geq 1}$ is summable.

Proof: By the reverse triangle inequality, we have for all $n \in \mathbf{Z}_{\geq 1}$

$$|z - w^n| \geq |w^n| - |z| = 1 - |z| > 0$$

so

$$0 \leq \frac{1}{n^2|z - w^n|} \leq \frac{1}{n^2(1 - |z|)} \text{ for all } n \in \mathbf{Z}_{\geq 1}.$$

Since $\left\{c \cdot \frac{1}{n^2}\right\}_{n \geq 1}$ is a summable sequence for all $c \in \mathbf{C}$, it follows from the comparison test that $\left\{\frac{1}{n^2|z - w^n|}\right\}_{n \geq 1}$ is summable. ‖

11.17 Example. Let $f(n) = \frac{(99.99)^n}{n!}$ for all $n \in \mathbf{N}$. Then

$$f(n+1) = \frac{(99.99)^{n+1}}{(n+1)!} = \frac{(99.99) \cdot (99.99)^n}{(n+1)n!} = \frac{99.99}{n+1} \cdot f(n).$$

If $n \geq 100$, then $n + 1 \geq 101$, so

$$f(n+1) = \frac{99.99}{n+1} \cdot f(n) \leq \frac{99.99}{101} f(n).$$

Hence,

$$\begin{aligned}
f(101) &\leq \left(\frac{99.99}{101}\right) f(100) \\
f(102) &\leq \left(\frac{99.99}{101}\right) \cdot f(101) \leq \left(\frac{99.99}{100}\right)^2 f(100) \\
f(103) &\leq \left(\frac{99.99}{101}\right) \cdot f(102) \leq \left(\frac{99.99}{100}\right)^3 f(100).
\end{aligned}$$

Hence, (by induction)

$$\begin{aligned}
f(100+n) &\leq \left(\frac{99.99}{101}\right)^n f(100) \\
&= \left(\frac{99.99}{101}\right)^{n+100} \left[\left(\frac{101}{99.99}\right)^{100} f(100)\right] \\
&= C \left(\frac{99.99}{101}\right)^{100+n}
\end{aligned}$$

where $C = \left(\frac{101}{99.99}\right)^{100} f(100)$; i.e., $f(j) \leq C\left(\frac{99.99}{101}\right)^j$ for all $j > 100$. Since the geometric series $\left\{\sum_{j=0}^{n}\left(\frac{99.99}{101}\right)^j\right\}$ converges, it follows from the comparison test that $\left\{\sum_{j=0}^{n}\frac{(99.99)^j}{j!}\right\}$ converges also.

11.18 Exercise. Determine whether or not the sequences below are summable:

(a) $\{(-1)^n\}$

(b) $\{(-1)^n + (-1)^{n+1}\}$

(c) $\{(-1)^n\} + \{(-1)^{n+1}\}$

(d) $\left\{\frac{n^2}{n^4+1}\right\}_{n\geq 1}$

(e) $\left\{1 - \frac{n}{n+1}\right\}_{n\geq 1}$

(f) $\left\{\frac{1}{n^3+\sqrt{n}}\right\}_{n\geq 1}$

(g) $\left\{\frac{n^2+n}{n^4+1}\right\}_{n\geq 1}$

(h) $\left\{\frac{3^n}{n!}\right\}$

11.19 Exercise. Give examples of the following, or explain why no such examples exist.

a) Two real sequences f and g such that f and g are not summable, but $f+g$ is summable.

b) Two real sequences f and g such that f and g are summable, but $f+g$ is not summable.

c) Two real sequences f and g such that $f(n) < g(n)$ for all $n \in \mathbf{N}$, and g is summable but f is not summable.

11.20 Theorem (Limit comparison test.) *Let f, g be sequences of positive numbers. Suppose that $\frac{f}{g}$ converges to a non-zero limit L. Then f is summable if and only if g is summable.*

Proof: We know that $L > 0$. Let $N = N_{\frac{f}{g} - \tilde{L}}$ be a precision function for $\frac{f}{g} - \tilde{L}$. Then

$$\left|\frac{f(n)}{g(n)} - L\right| \leq \frac{L}{2} \text{ for all } n \geq N\left(\frac{L}{2}\right);$$

i.e.,

$$\frac{L}{2} \leq \frac{f(n)}{g(n)} \leq \frac{3L}{2} \text{ for all } n \geq N\left(\frac{L}{2}\right).$$

If g is summable, then $\frac{3L}{2}g$ is summable, and since $f(n) \leq \frac{3L}{2}g(n)$ for all $n \geq N\left(\frac{L}{2}\right)$, it follows from the comparison test that f is summable. If g is not summable, then since $g(n) \leq \frac{2}{L}f(n)$ for all $n \geq N(\frac{l}{2})$ it follows that $\frac{2}{L}f$ is not summable, and hence f is not summable. |||

11.21 Example. Is $\left\{\frac{n^2+5n+1}{6n^3+3n-2}\right\}_{n\geq 1}$ summable? Let $a_n = \frac{n^2+5n+1}{6n^3+3n-2}$. Note that $a_n > 0$ for all $n \in \mathbf{Z}_{\geq 1}$. For large n, a_n is "like" $\frac{n^2}{6n^3} = \frac{1}{6n}$, so I'll compare this series with $\left\{\frac{1}{n}\right\}_{n\geq 1}$. Let $b_n = \frac{1}{n}$ for all $n \in \mathbf{Z}_{\geq 1}$. Then

$$\frac{a_n}{b_n} = \frac{n^3+5n^2+n}{6n^3+3n-2} = \frac{1+\frac{5}{n}+\frac{1}{n^3}}{6+\frac{3}{n^2}-\frac{2}{n^3}},$$

so

$$\left\{\frac{a_n}{b_n}\right\}_{n\geq 1} = \left\{\frac{1+\frac{5}{n}+\frac{1}{n^3}}{6+\frac{3}{n^2}-\frac{2}{n^3}}\right\}_{n\geq 1} \to \frac{1+0+0}{6+0+0} = \frac{1}{6} \neq 0.$$

Since $\{b_n\}_{n\geq 1} = \left\{\frac{1}{n}\right\}_{n\geq 1}$ is not summable, $\left\{\frac{n^2+5n+1}{6n^3+3n-2}\right\}_{n\geq 1}$ is also not summable.

11.22 Exercise. Determine whether or not the sequences below are summable.

a) $\left\{\dfrac{n^2+3n+2}{n^4+n+1}\right\}_{n\geq 1}$.

b) $\left\{\dfrac{n+n^2}{n^3+n+1}\right\}_{n\geq 1}$.

11.23 Theorem (Ratio test.) *Let $\{a_n\}$ be a sequence of positive numbers. Suppose the $\left\{\dfrac{a_{n+1}}{a_n}\right\}$ converges, and $\lim\left\{\dfrac{a_{n+1}}{a_n}\right\} = R$. Then, if $R < 1$, $\{a_n\}$ is summable. If $R > 1$, $\{a_n\}$ is not summable. (If $R = 1$, the theorem makes no assertion.)*

Proof: Suppose $\left\{\dfrac{a_{n+1}}{a_n}\right\} \to R$.

Case 1: $R < 1$. Let N be a precision function for $\left\{\dfrac{a_{n+1}}{a_n} - R\right\}$. Then for all $n \in \mathbf{N}$,

$$\begin{aligned} n \geq N\left(\frac{1-R}{2}\right) &\implies \left|\frac{a_{n+1}}{a_n} - R\right| < \frac{1-R}{2} \\ &\implies \frac{a_{n+1}}{a_n} < R + \frac{1-R}{2} = \frac{R+1}{2}. \end{aligned}$$

Write $M = N\left(\dfrac{1-R}{2}\right)$ and $S = \dfrac{1+R}{2}$, so $(0 < S < 1)$. Then

$$n \geq M \implies a_{n+1} \leq S \cdot a_n,$$

so

$$\begin{aligned} a_M &\leq S^0 a_M \\ a_{M+1} &\leq S a_M \\ a_{M+2} &\leq S \cdot a_{M+1} \leq S^2 a_M \\ a_{M+3} &\leq S \cdot a_{M+2} \leq S^3 a_M, \end{aligned}$$

and (by an induction argument which I omit)

$$a_{M+k} \leq S^k a_M \text{ for all } k \in \mathbf{N},$$

or

$$a_{M+k} \leq S^{M+k}(a_M S^{-M}) \text{ for all } k \in \mathbf{N},$$

or

$$a_n \leq S^n(a_M S^{-M}) \text{ for all } n \in \mathbf{Z}_{\geq M}.$$

Since $\{S^n\}$ is a summable geometric series, it follows from the comparison test that $\{a_n\}$ is also summable.

Case 2: $(R > 1)$. As before, let N be a precision function for $\left\{\frac{a_{n+1}}{a_n} - R\right\}$. Then for all $n \in \mathbf{N}$,

$$\begin{aligned} n \geq N\left(\frac{R-1}{2}\right) &\implies \left|\frac{a_{n+1}}{a_n} - R\right| < \frac{R-1}{2} \\ &\implies \frac{a_{n+1}}{a_n} > R - \left(\frac{R-1}{2}\right) = \frac{R+1}{2} > 1 \\ &\implies a_{n+1} > a_n. \end{aligned}$$

Hence $\{a_n\}$ is not a null sequence. So $\{a_n\}$ is not summable. |||

11.24 Warning. The ratio test does not say that if $\frac{a_{n+1}}{a_n} < 1$ for all n, then $\{a_n\}$ is summable. If $a_n = \frac{1}{n}$ for $n \in \mathbf{Z}_{\geq 1}$, then $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} < 1$ for all n but $\{a_n\}$ is not summable. (In this case, $\lim\left\{\frac{a_{n+1}}{a_n}\right\} = 1$, and the ratio test does not apply.)

If $b_n = \frac{1}{n^2}$ for all $n \in \mathbf{Z}_{\geq 1}$, then $\frac{b_{n+1}}{b_n} = \left(\frac{n^2}{(n+1)^2}\right)$ for all n and hence $\lim\left\{\frac{b_{n+1}}{b_n}\right\} = 1$, and $\{b_n\}$ is summable. These examples show that when $\lim\left\{\frac{a_{n+1}}{a_n}\right\} = 1$ the ratio test gives no useful information.

11.25 Remark. If, in applying the ratio test, you find that $\frac{a_{n+1}}{a_n} \geq 1$ for all large n, you can conclude that $\sum\{a_n\}$ diverges (even if $\lim\left\{\frac{a_{n+1}}{a_n}\right\}$ does not exist), since this condition shows that $\{a_n\}$ is not a null sequence.

11.26 Example. Let t be a positive number and let $a_n = \dfrac{(3n)!t^n}{(n!)^3}$. We apply the ratio test to the series $\sum\{a_n\}$.

$$\frac{a_{n+1}}{a_n} = \frac{(3(n+1))!t^{n+1}(n!)^3}{[(n+1)!]^3(3n)!t^n}.$$

Note that

$$\begin{aligned} (3(n+1))! &= (3n+3)! = (3n+3)(3n+2)! = (3n+3)(3n+2)(3n+1)! \\ &= (3n+3)(3n+2)(3n+1)(3n)!. \end{aligned}$$

Hence

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(3n)!t\cdot(3n+3)(3n+2)(3n+1)}{(3n)!}\left(\frac{n!}{(n+1)!}\right)^3 \\ &= t\left(\frac{3n+3}{n+1}\right)\left(\frac{3n+2}{n+1}\right)\left(\frac{3n+1}{n+1}\right) \\ &= t\cdot 3\cdot\left(\frac{3+\frac{2}{n}}{1+\frac{1}{n}}\right)\left(\frac{3+\frac{1}{n}}{1+\frac{1}{n}}\right). \end{aligned}$$

From this we see that $\left\{\dfrac{a_{n+1}}{a_n}\right\} \to 27t$. The ratio test says that if $27t < 1$ (i.e., if $t < \frac{1}{27}$), then $\left\{\dfrac{(3n)!t^n}{(n!)^3}\right\}$ is summable, and if $t > \frac{1}{27}$, then the sequence is not summable.

Can we figure out what happens in the case $t = \dfrac{1}{27}$? For $t = \dfrac{1}{27}$, our formula above gives us

$$\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{2}{3n}\right)\left(1+\frac{1}{3n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} = \frac{1+\frac{1}{n}+\frac{2}{9n^2}}{\left(1+\frac{1}{n}\right)^2} > \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2} = \frac{n}{n+1};$$

i.e., $a_{n+1} > \dfrac{n}{n+1}a_n$ for $n \geq 1$. Thus,

$$\begin{aligned} a_2 &\geq \frac{1}{2}\cdot a_1 \\ a_3 &\geq \frac{2}{3}a_2 \geq \frac{2}{3}\cdot\frac{1}{2}a_1 = \frac{1}{3}a_1 \\ a_4 &\geq \frac{3}{4}a_3 \geq \frac{3}{4}\cdot\frac{1}{3}a_1 = \frac{1}{4}a_1 \\ a_5 &\geq \frac{4}{5}a_4 \geq \frac{4}{5}\cdot\frac{1}{4}a_1 = \frac{1}{5}a_1 \end{aligned}$$

and (by induction),

$$a_n \geq \frac{1}{n} a_1 \text{ for } n \geq 1.$$

Since $\left\{\frac{a_1}{n}\right\}_{n\geq 1}$ is not summable, it follows that $\{a_n\}$ is not summable for $t = \frac{1}{27}$.

11.27 Example. Let $b_n = \frac{(n!)^2 4^n}{(2n)!}$ for all $n \in \mathbf{N}$. I'll apply ratio test to $\sum\{b_n\}$. For all $n \in \mathbf{N}$,

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{(n+1)!^2 4^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2 4^n} \\ &= \frac{(n+1)^2 \cdot 4}{(2n+1)(2n+2)} = \frac{2n+2}{2n+1} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}} \end{aligned}$$

Hence $\left\{\frac{b_{n+1}}{b_n}\right\} \to 1$ and the ratio test does not apply. But since $\frac{2n+2}{2n+1} > 1$ for all n, I conclude that $\{b_n\}$ is an increasing sequence and hence $\sum\{b_n\}$ diverges.

11.28 Exercise. For each of the series below, determine for which $x \in [0, \infty)$ the series converges.

a) $\sum\left\{\frac{x^n}{n!}\right\}$

b) $\sum\left\{\frac{x^{2n}}{(2n)!}\right\}$

c) $\sum\left\{\frac{3^n x^n}{n^2}\right\}_{n\geq 1}$

d) $\sum\left\{\frac{x^n}{2^n\sqrt{n}}\right\}_{n\geq 1}$

e) $\sum\{nx^n\}_{n\geq 1}$

f) $\sum\{n!x^n\}$

g) $\sum\left\{\frac{(n!)^2 x^n}{(2n)!}\right\}$ [For this series, there is one $x \in [0, \infty)$ for which you don't need to answer the question.]

11.3 Alternating Series

11.29 Definition (Alternating series.) Series of the form $\sum\{(-1)^n a_n\}$ or $\sum\{(-1)^{n+1}a_n\}$ where $a_j \geq 0$ for all j are called *alternating series.*

11.30 Theorem (Alternating series test.) *Let f be a decreasing sequence of positive numbers such that $\{f(n)\} \to 0$. Then $\{(-1)^n f(n)\}$ is summable. Moreover,*

$$\sum_{j=0}^{2m+1}(-1)^j f(j) \leq \sum_{j=0}^{\infty}(-1)^j f(j) \leq \sum_{j=0}^{2n}(-1)^j f(j)$$

and

$$\left|\sum_{j=0}^{n}(-1)^j f(j) - \sum_{j=0}^{\infty}(-1)^j f(j)\right| \leq f(n+1)$$

for all $m, n \in \mathbf{N}$.

Proof: Let $S_n = \sum_{j=0}^{n}(-1)^j f(j)$. For all $n \in \mathbf{N}$,

$$S_{2(n+1)} = S_{2n+2} = S_{2n} - f(2n+1) + f(2n+2) \leq S_{2n}$$

and

$$S_{2(n+1)+1} = S_{2n+1} + f(2n+2) - f(2n+3) \geq S_{2n+1}.$$

Thus $\{S_{2n}\}$ is decreasing and $\{S_{2n+1}\}$ is increasing. Also, for all $n \in \mathbf{N}$,

$$S_1 \leq S_{2n+1} = S_{2n} - f(2n+1) \leq S_{2n}$$

so $\{S_{2n}\}$ is bounded below by S_1, and

$$S_{2n+1} = S_{2n} - f(2n+1) \leq S_{2n} \leq S_0$$

so $\{S_{2n+1}\}$ is bounded above by S_0.

It follows that there exist real numbers L and M such that

$$\begin{aligned}&\{S_{2n}\} \to L \text{ and } S_{2n} \geq L \text{ for all } n \in \mathbf{N}\\ &\{S_{2n+1}\} \to M \text{ and } S_{2n+1} \leq M \text{ for all } n \in \mathbf{N}.\end{aligned}$$

Now

$$\begin{aligned}L - M &= \lim\{S_{2n}\} - \lim\{S_{2n+1}\} = \lim\{S_{2n} - S_{2n+1}\}\\ &= \lim\{f(2n+1)\} = 0,\end{aligned}$$

so $L = M$.

It follows from the next lemma that $\{S_n\} \to L$; i.e.,

$$M = L = \lim S_n = \sum_{n=0}^{\infty} (-1)^n f(n).$$

Since for all $n \in \mathbf{N}$

$$S_{2n+1} \leq L \leq S_{2n},$$

we have

$$|L - S_{2n}| \leq S_{2n} - S_{2n+1} = f(2n+1)$$

and since

$$\begin{aligned} S_{2n+1} \leq L &\leq S_{2n+2} \\ |L - S_{2n+1}| &\leq S_{2n+2} - S_{2n+1} = f(2n+2). \end{aligned}$$

Thus, in all cases, $|L - S_n| < f(n+1)$; i.e., $\sum_{j=0}^{n} (-1)^j f(j)$ approximates $\sum_{j=0}^{\infty} (-1)^j f(j)$ with an error of no more than $f(n+1)$. ∥

11.31 Lemma. *Let $\{a_n\}$ be a real sequence and let $L \in \mathbf{R}$. Suppose $\{a_{2n}\} \to L$ and $\{a_{2n+1}\} \to L$. Then $\{a_n\} \to L$.*

Proof: Let N be a precision function for $\{a_{2n} - L\}$ and let M be a precision function for $\{a_{2n+1} - L\}$. For all $\varepsilon \in \mathbf{R}^+$, define

$$N_{a-\tilde{L}}(\varepsilon) = \max\left(2N(\varepsilon), 2M(\varepsilon) + 1\right).$$

I claim $N_{a-\tilde{L}}$ is a precision function for $a - \tilde{L}$, and hence $a \to L$. Let $n \in \mathbf{N}$.

Case 1: n is even. Suppose n is even. Say $n = 2k$ where $k \in \mathbf{N}$. Then

$$\begin{aligned} (n \geq N_{a-\tilde{L}}(\varepsilon) &\implies 2k \geq N_{a-\tilde{L}}(\varepsilon) \geq 2N(\varepsilon) \\ &\implies k > N(\varepsilon) \implies |a_{2k} - L| < \varepsilon \\ &\implies |a_n - L| < \varepsilon, \end{aligned}$$

Case 2: n is odd. Suppose n is odd. Say $n = 2k + 1$ where $k \in \mathbf{N}$. Then

$$\begin{aligned} (n \geq N_{a-\tilde{L}}(\varepsilon) &\implies 2k + 1 \geq N_{a-\tilde{L}}(\varepsilon) \geq 2M(\varepsilon) + 1 \\ &\implies k \geq M(\varepsilon) \implies |a_{2k+1} - L| < \varepsilon \\ &\implies |a_n - L| < \varepsilon. \end{aligned}$$

Hence, in all cases,

$$n \geq N_{a-\tilde{L}}(\varepsilon) \implies |a_n - L| < \varepsilon. \; \|$$

11.32 Remark. The alternating series test has obvious generalizations for series such as

$$\sum\{(-1)^{j+1}f(j)\} \text{ or } \sum\{(-1)^j f(j)\}_{j\geq k},$$

and we will use these generalizations.

11.33 Example. If $0 \leq t \leq 1$, then

$$\left\{\frac{t^{2n}}{(2n)!}\right\} \text{ and } \left\{\frac{t^{2n+1}}{(2n+1)!}\right\}$$

are decreasing positive null sequences, so

$$\left\{\frac{(-1)^n t^{2n}}{(2n)!}\right\} \text{ and } \left\{\frac{(-1)^n t^{2n+1}}{(2n+1)!}\right\}$$

are summable; i.e.,

$$\left\{\sum_{j=0}^{n}\frac{(-1)^j t^{2j}}{(2j)!}\right\} \text{ and } \left\{\sum_{j=0}^{n}\frac{(-1)^j t^{2j+1}}{(2j+1)!}\right\} \text{ converge.}$$

(These are the sequences we called $\{C_n(t)\}$ and $\{S_n(t)\}$ in example 10.45.)

Also, $\displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j\left(\frac{1}{10}\right)^{2j}}{(2j)!} = 1 - \frac{1}{200} + \frac{1}{240000}$, with an error smaller than $\dfrac{1}{720000000}$. My calculator says

$$\cos(.1) = 0.995004165$$

and

$$1 - \frac{1}{200} + \frac{1}{240000} = 0.995004166.$$

11.34 Entertainment. Since $\left\{\dfrac{t^n}{n}\right\}_{n\geq 1}$ is a decreasing positive null sequence for $0 \leq t \leq 1$, it follows that $\displaystyle\sum\left\{\frac{(-1)^{n-1}t^n}{n}\right\}_{n\geq 1}$ converges for

$0 \le t \le 1$. We will now explicitly calculate the limit of this series using a few ideas that are not justified by results proved in this course. We know that for all $x \in \mathbf{R} \setminus \{-1\}$, and all $n \in \mathbf{N}$,

$$1 - x + x^2 - x^3 + \cdots + (-x)^{n-1} = \frac{1-(-x)^n}{1-(-x)} = \frac{1}{1+x} + (-1)^{n+1}\frac{x^n}{1+x}.$$

Hence, for all $t > -1$,

$$\int_0^t 1 - x + x^2 + \cdots + (-x)^{n-1} dx = \int_0^t \frac{1}{1+x}dx + (-1)^{n+1}\int_0^t \frac{x^n}{1+x}dx;$$

i.e.,

$$x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n-1}x^n}{n}\Big|_0^t = \ln(1+x)\Big|_0^t + (-1)^{n+1}\int_0^t \frac{x^n}{1+x}dx.$$

Thus

$$t - \frac{t^2}{2} + \frac{t^3}{3} + \cdots + \frac{(-1)^{n-1}t^n}{n} = \ln(1+t) + (-1)^{n+1}\int_0^t \frac{x^n}{1+x}dx.$$

Hence

$$\left| t - \frac{t^2}{2} + \frac{t^3}{3} + \cdots + \frac{(-1)^{n-1}t^n}{n} - \ln(1+t) \right| = \left| \int_0^t \frac{x^n}{1+x}dx \right|$$

for all $t > -1$.

If we can show that $\left\{ \int_0^t \frac{x^n}{1+x}dx \right\}$ is a null sequence, it follows that

$$\left\{ t - \frac{t^2}{2} + \frac{t^3}{3} + \cdots + \frac{(-1)^{n-1}t^n}{n} \right\} \to \ln(1+t),$$

or in other words,

$$\ln(1+t) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}t^j}{j}. \tag{11.35}$$

I claim $\left\{ \int_0^t \frac{x^n}{1+x}dx \right\}$ is a null sequence for $-1 < t \le 1$ and hence (11.35) holds for $-1 < t \le 1$. In particular,

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

First suppose $t \geq 0$, then $\dfrac{1}{1+x}x^n \leq x^n$ for $0 \leq x \leq t$, so

$$0 = \int_0^t \frac{1}{1+x} \cdot x^n dx \leq \int_0^t x^n dx = \frac{x^{n+1}}{n+1}\bigg|_0^t = \frac{t^{n+1}}{n+1}.$$

Since $\left\{\dfrac{t^{n+1}}{n+1}\right\}$ is a null sequence for $0 \leq t \leq 1$, it follows from the comparison test that $\left\{\displaystyle\int_0^t \frac{x^n}{1+x} dx\right\}$ is a null sequence for $0 \leq t \leq 1$. Now suppose $-1 < t < 0$. Then

$$\frac{1}{1+x} \leq \frac{1}{1+t} \text{ for } t \leq x \leq 0,$$

so $\dfrac{|x|^n}{1+x} \leq \dfrac{|x|^n}{1+t}$ and

$$\begin{aligned}
\left|\int_0^t \frac{x^n}{1+x} dx\right| &= \int_t^0 \frac{|x|^n}{1+x} dx \leq \int_t^0 \frac{|x|^n}{1+t} dx \\
&= \frac{1}{1+t} \int_t^0 |x|^n dx = \frac{1}{1+t} \int_0^{|t|} x^n dx \\
&= \frac{1}{1+t} \cdot \frac{|t|^{n+1}}{n+1}.
\end{aligned}$$

If $-1 < t < 0$, then $\left\{\dfrac{1}{1+t} \cdot \dfrac{|t|^{n+1}}{n+1}\right\}$ is a null sequence, so $\left\{\displaystyle\int_0^t \frac{x^n}{1+x} dx\right\}$ is a null sequence. $\|$

11.36 Entertainment. By starting with the formula

$$1 - x^2 + x^4 - x^6 + \cdots + (-x^2)^{n-1} = \frac{1 - (-x^2)^n}{1 - (-x^2)}$$

for all $x \in \mathbf{R}$ and using the ideas from the last example, show that

$$\sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)} = \arctan(x) \text{ for all } x \in [-1, 1]. \tag{11.37}$$

Conclude that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

11.38 Exercise. Determine whether or not the following series converge.

a) $\sum\left\{(-1)^n \dfrac{n+1}{n^2}\right\}_{n\geq 1}$

b) $\sum\left\{(-1)^n \dfrac{n+1}{n}\right\}_{n\geq 1}$

c) $\sum\left\{\dfrac{(-1)^n t^{2n}}{(2n)!}\right\}$ (assume here $-1 \leq t \leq 1$).

11.4 Absolute Convergence

11.39 Definition (Absolute Convergence.) Let f be a complex sequence. We say that f is *absolutely summable* if and only if $|f|$ is summable; i.e., if and only if $\{\sum_{j=0}^{n} |f(j)|\}$ converges. In this case, we also say that the series $\sum f$ is *absolutely convergent.*

11.40 Example. $\sum\left\{\dfrac{(-1)^n}{n}\right\}_{n\geq 1}$ is convergent, but is not absolutely convergent.

11.41 Theorem. *Let f be a complex sequence. If $\sum f$ is absolutely convergent, then $\sum f$ is convergent.*

Proof:

Case 1: Suppose $f(n)$ is real for all $n \in \mathbf{N}$, and that $\sum |f|$ converges. Then

$$0 \leq f(n) + |f(n)| \leq |f(n)| + |f(n)| = 2|f(n)|$$

for all $n \in \mathbf{N}$, so by the comparison test, $\sum(f + |f|)$ converges. Then $\sum(f + |f|) - \sum |f|$, being the difference of two convergent sequences, is convergent; i.e., $\sum f$ converges.

Case 2: Suppose f is an arbitrary absolutely convergent complex series. We know that for all $n \in \mathbf{N}$,

$$0 \leq |\mathrm{Re}(f)(n)| \leq |f(n)|$$

and

$$0 \leq |\mathrm{Im}(f)(n)| \leq |f(n)|,$$

so by the comparison test, $\sum |\mathrm{Re}(f)|$ and $\sum |\mathrm{Im}(f)|$ are convergent, and by Case 1, $\sum(\mathrm{Re}(f))$ and $\sum(\mathrm{Im}(f))$ are convergent. It follows that $\sum(\mathrm{Re}(f)) + i\sum(\mathrm{Im}(f)) = \sum f$ is convergent. ‖

11.42 Example. Let z be a non-zero complex number. Let

$$\{C_n(z)\} = \sum\{c_n(z)\} = \left\{\sum_{j=0}^{n} \frac{z^{2j}(-1)^j}{(2j)!}\right\}.$$

I claim $\sum\{c_n\}$ is absolutely convergent (and hence convergent). We have

$$|c_n(z)| = \frac{|z|^{2n}}{(2n)!}.$$

We have

$$\left\{\frac{|c_{n+1}(z)|}{|c_n(z)|}\right\} = \left\{\frac{|z|^{2n+2}}{(2n+2)!}\frac{(2n)!}{|z|^{2n}}\right\} = \left\{\frac{|z|^2}{(2n+1)(2n+2)}\right\} \to 0 < 1$$

so by the ratio test, $\sum\{|c_n(z)|\}$ converges. Hence $\sum\{c_n(z)\}$ is absolutely convergent, and hence it is convergent. Clearly $\{C_n(0)\} \to 1$, so $\{C_n(z)\}$ converges for all $z \in \mathbf{C}$. In the exercises you will show that $\sum\left\{\frac{(-1)^j z^{2j+1}}{(2j+1)!}\right\}$ is also convergent for all $z \in \mathbf{C}$.

Motivated by the results of section 10.3, we make the following definitions:

11.43 Definition (sin and cos.) For all $z \in \mathbf{C}$, we define

$$\begin{aligned}\cos(z) &= \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j)!}.\\ \sin(z) &= \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!}.\end{aligned}$$

11.44 Remark. It is clear from the definition that

$$\sin(0) = 0 \text{ and } \cos(0) = 1.$$
$$\sin(-z) = -\sin(z) \text{ for all } z \in \mathbf{C}.$$
$$\cos(-z) = \cos(z) \text{ for all } z \in \mathbf{C}.$$

For all $n \in \mathbf{N}$, $z \in \mathbf{C}$, let

$$\begin{aligned} C_n(z) &= \sum_{j=0}^{n} \frac{(-1)^j z^{2j}}{(2j)!}, \\ S_n(z) &= \sum_{j=0}^{n} \frac{(-1)^j z^{2j+1}}{(2j+1)!}. \end{aligned}$$

Then

$$\begin{aligned} S_n'(z) &= \sum_{j=0}^{n} \frac{(-1)^j (2j+1) z^{2j}}{(2j+1)!} \\ &= \sum_{j=0}^{n} \frac{(-1)^j z^{2j}}{(2j)!} = C_n(z). \end{aligned}$$

I would now like to be able to say that for all $z \in \mathbf{C}$,

$$\begin{aligned} \{S_n(z)\} \to S(z) &\implies \{S_n'(z)\} \to S'(z) \\ &\implies \{C_n(z)\} \to S'(z) \\ &\implies S'(z) = C(z) \text{ (since } \{C_n\} \to C); \end{aligned}$$

i.e., I would like to have a theorem that says

$$\{f_n(z)\} \to f(z) \implies \{f_n'(z)\} \to f'(z).$$

However, the next example shows that this hoped for theorem is not true.

11.45 Example. Let $f_n(z) = \dfrac{z}{1+nz^2}$ for all $z \in \mathbf{C}$, $n \in \mathbf{Z}_{\geq 1}$. Then for all $z \in \mathbf{C}\backslash\{0\}$,

$$\{f_n(z)\} = \frac{z}{n} \frac{1}{\left(\frac{1}{n} + z^2\right)} \to 0 \cdot \frac{1}{0 + z^2} = 0,$$

and

$$\{f_n(0)\} = \{0\} \to 0,$$

so

$$f_n(z) \to \tilde{0}(z) \text{ for all } z \in \mathbf{C}.$$

Now $f_n'(z) = \dfrac{(1+nz^2) - 2nz^2}{(1+nz^2)^2} = \dfrac{1-nz^2}{(1+nz^2)^2}$. So $f_n'(0) = 1$ for all n, and thus $\{f_n'(0)\} \to 1 \neq \tilde{0}'(0)$. Eventually we will show that $\sin' = \cos$ and $\cos' = -\sin$, but it will require some work.

11.46 Warning. Defining sine and cosine in terms of infinite series can be dangerous to the well being of the definer. In 1933 Edmund Landau was forced to resign from his position at the University of Göttingen as a result of a Nazi-organized boycott of his lectures. Among other things, it was claimed that Landau's definitions of sine and cosine in terms of power series was "un-German", and that the definitions lacked "sense and meaning" [33, pp 226–227].

11.47 Exercise. Show that $\sum \left\{ \dfrac{(-1)^n z^{2n+1}}{(2n+1)!} \right\}$ converges for all $z \in \mathbf{C}$.

11.48 Exercise.

a) Does the series $\sum \left\{ \dfrac{\left(\frac{3}{5} + \frac{4i}{5}\right)^n}{n^2} \right\}_{n \geq 1}$ converge?

b) Does the sequence $\left\{ \displaystyle\sum_{j=1}^{4n} \frac{i^j}{j} \right\}_{n \geq 1}$ converge?

11.49 Exercise.

a) For what complex numbers z does $\sum\{nz^n\}$ converge?

b) For what complex numbers z does $\sum\{z^{(n^2)}\}$ converge?

11.50 Note. The harmonic series was shown to be unbounded by Nicole Oresme c. 1360 [31, p437]. However, many 17th and 18th century mathematicians believed that (in our terminology) every null sequence is summable. Jacob Bernoulli rediscovered Oresme's result in 1687, and reported that it contradicted his earlier belief that an infinite series whose last term vanishes must be finite[31, p 437]. As late as 1770, Lagrange said that a series represents a number if its nth term approaches 0 [31, p 464].

The ratio test was stated by Jean D'Alembert in 1768, and by Edward Waring in 1776[31, p 465]. D'Alembert knew that the ratio test guaranteed absolute convergence.

The alternating series test appears in a letter from Leibniz to Jacob Bernoulli written in 1713[31, p461].

The series (11.35) for $\ln(1+t)$ is called *Mercator's formula* after Nicolaus Mercator who published it in 1668. It was discovered earlier by Newton in 1664 when he was an undergraduate at Cambridge. After Newton read Mercator's book, he quickly wrote down some of his own ideas (which were much more general than Mercator's) and allowed his notes to be circulated, but not published. Newton used the logarithm formula to calculate $\ln(1.1)$ to 68 decimals (of which the 28th and 43rd were wrong), but a few years later, he redid the calculation and corrected the errors.

See [22, chapter 2] for a discussion of Newton's work on series.

The series representation for arctan (11.37) is called *Gregory's formula* after John Gregory (1638-1675) or *Leibniz's formula* after Gottfried Leibniz (1646-1716). However, it was known to sixteenth century Indian mathematicians who credited it to Madhava (c. 1340-1425). The Indian version was

$$\theta = \frac{\sin\theta}{\cos\theta} - \frac{1}{3}\frac{\sin^3\theta}{\cos^3\theta} + \frac{1}{5}\frac{\sin^5\theta}{\cos^5\theta}\cdots.$$

(See[30, p292].)