Chapter 10

The Derivative

10.1 Derivatives of Complex Functions

You are familiar with derivatives of functions from $\mathbf{R}$ to $\mathbf{R}$, and with the motivation of the definition of derivative as the slope of the tangent to a curve. For complex functions, the geometrical motivation is missing, but the definition is formally the same as the definition for derivatives of real functions.

10.1 Definition (Derivative.) Let f be a complex valued function with $\text{dom}(f) \subset \mathbf{C}$, let a be a point such that $a \in \text{dom}(f)$, and a is a limit point of $\text{dom}(f)$. We say f is *differentiable at a* if the limit

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists. In this case, we denote this limit by $f'(a)$ and call $f'(a)$ the *derivative of f at a*.

By the definition of limit, we can say that f is differentiable at a if $a \in \text{dom}(f)$, and a is a limit point of $\text{dom}(f)$ and there exists a function $D_a f : \text{dom}(f) \to \mathbf{C}$ such that $D_a f$ is continuous at a, and such that

$$D_a f(z) = \frac{f(z) - f(a)}{z - a} \text{ for all } z \in \text{dom}(f) \setminus \{a\}, \tag{10.2}$$

and in this case $f'(a)$ is equal to $D_a f(a)$.

It is sometimes useful to rephrase condition (10.2) as follows: f is differentiable at a if $a \in \text{dom}(f)$, a is a limit point of $\text{dom}(f)$, and there is a function $D_a f \colon \text{dom} f \to \mathbf{C}$ such that $D_a f$ is continuous at a, and

$$f(z) = f(a) + (z-a)D_a f(z) \text{ for all } z \in \text{dom}(f). \tag{10.3}$$

In this case, $f'(a) = D_a f(a)$.

10.4 Remark. It follows immediately from (10.3) that if f is differentiable at a, then f is continuous at a.

10.5 Example. Let $f \colon \mathbf{C} \to \mathbf{C}$ be given by

$$f \colon z \mapsto z^2,$$

and let $a \in \mathbf{C}$. Then for all $z \neq a$,

$$\frac{f(z) - f(a)}{z-a} = \frac{z^2 - a^2}{z-a} = z + a.$$

If we define $D_a f \colon \mathbf{C} \to \mathbf{C}$ by

$$D_a f(z) = z + a \text{ for all } z \in \mathbf{C},$$

then $D_a f$ is continuous at a, so f is differentiable at a and

$$f'(a) = D_a f(a) = a + a = 2a \text{ for all } a \in \mathbf{C}.$$

We could also write this calculation as

$$\lim_{z \to a} \frac{f(z) - f(a)}{z-a} = \lim_{z \to a} \frac{z^2 - a^2}{z-a} = \lim_{z \to a} z + a = a + a = 2a.$$

Hence f is differentiable at a and $f'(a) = 2a$ for all $a \in \mathbf{C}$.

10.6 Example. Let $v(z) = \dfrac{1}{z}$ for $z \in \mathbf{C} \backslash \{0\}$ and let $a \in \mathbf{C} \backslash \{0\}$. Then for all $z \in \mathbf{C} \backslash \{a\}$

$$\frac{v(z) - v(a)}{z-a} = \frac{\frac{1}{z} - \frac{1}{a}}{z-a} = \frac{a-z}{za(z-a)} = -\frac{1}{za}.$$

Let $D_a v\colon \mathbf{C}\backslash\{0\} \to \mathbf{C}$ be defined by

$$D_a v(z) = -\frac{1}{za} \text{ for all } z \in \mathbf{C}\backslash\{0\}.$$

Then $D_a v$ is continuous at a, so v is differentiable at a, and

$$v'(a) = D_a v(a) = -\frac{1}{a^2}$$

for all $a \in \mathbf{C}\backslash\{0\}$.

10.7 Warning. The function $D_a f$ should not be confused with f'. In the example above

$$D_a v(z) = -\frac{1}{za}, \quad v'(z) = \frac{-1}{z^2}.$$

Also it is not good form to say

$$D_a f(z) = \frac{f(z) - f(a)}{z - a} \tag{10.8}$$

without specifying the condition "for $z \neq a$," since someone reading (10.8) would assume $D_a f$ is undefined at a.

10.9 Example. Let $f(z) = z^*$ for all $z \in \mathbf{C}$, and let $a \in \mathbf{C}$. Let

$$D_a f(z) = \frac{f(z) - f(a)}{z - a} = \frac{z^* - a^*}{z - a} \text{ for all } z \in \mathbf{C} \setminus \{a\}.$$

I claim that $D_a f$ does not have a limit at a, and hence f is a nowhere differentiable function.

Let

$$\{a_n\}_{n\geq 1} = \left\{a + \frac{1}{n}\right\}_{n\geq 1}, \{b_n\}_{n\geq 1} = \left\{a + \frac{i}{n}\right\}_{n\geq 1}.$$

Then $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are sequences in $\text{dom}(f)\backslash\{a\}$ both of which converge to a. For all $n \in \mathbf{Z}_{\geq 1}$,

$$\begin{aligned} D_a f(a_n) &= \frac{\left(a + \frac{1}{n}\right)^* - a^*}{a + \frac{1}{n} - a} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1, \\ D_a f(b_n) &= \frac{\left(a + \frac{i}{n}\right)^* - a^*}{a + \frac{i}{n} - a} = \frac{\frac{-i}{n}}{\frac{i}{n}} = -1, \end{aligned}$$

so $\{D_a f(a_n)\}_{n\geq 1} \to 1$ and $\{D_a f(b_n)\}_{n\geq 1} \to -1$, and hence $D_a f$ does not have a limit at a.

10.10 Exercise. Investigate the following functions for differentiability at an arbitrary point $a \in \mathbf{C}$. Calculate the derivatives of any differentiable functions.

a) $f(z) = Az + B$ $\qquad$ A, B are given complex numbers.

b) $g(z) = \dfrac{1}{(z+i)^2}$ $\qquad$ $z \in \mathbf{C}\backslash\{-i\}$.

c) $h(z) = \mathrm{Re}(z)$, i.e. $h(x+iy) = x$.

10.11 Theorem (Sum theorem for differentiable functions.) *Let f, g be complex functions, and suppose f and g are differentiable at $a \in \mathbf{C}$. Suppose a is a limit point of* $\mathrm{dom}(f) \cap \mathrm{dom}(g)$*. Then $f+g$ is differentiable at a and $(f+g)'(a) = f'(a) + g'(a)$.*

Proof: Since f, g are differentiable at a, there are functions $D_a f\colon \mathrm{dom}(f) \to \mathbf{C}$, $D_a g\colon \mathrm{dom}(g) \to g$ such that $D_a f$, $D_a g$ are continuous at a, and

$$\begin{aligned} f(z) &= f(a) + (z-a)D_a f(z) \text{ for all } z \in \mathrm{dom}(f) \\ g(z) &= g(a) + (z-a)D_a g(z) \text{ for all } z \in \mathrm{dom}(g). \end{aligned}$$

It follows that

$$(f+g)(z) = (f+g)(a) + (z-a)[D_a f(z) + D_a g(z)] \text{ for all } z \in \mathrm{dom}(f+g)$$

and $D_a f + D_a g$ is continuous at a.

We can let $D_a(f+g) = D_a f + D_a g$ and we see $f+g$ is differentiable at a and

$$(f+g)'(a) = (D_a f + D_a g)(a) = D_a f(a) + D_a g(a) = f'(a) + g'(a). \ \|$$

10.12 Theorem. *Let f be a complex function and let $c \in \mathbf{C}$. If f is differentiable at a, then cf is differentiable at a and $(cf)'(a) = c \cdot f'(a)$.*

Proof: The proof is left to you. $\|$

10.13 Theorem (Chain Rule.) *Let f, g be complex functions, and let $a \in \mathbf{C}$. Suppose f is differentiable at a, and g is differentiable at $f(a)$, and that a is a limit point of* $\mathrm{dom}(g \circ f)$*. Then the composition $(g \circ f)$ is differentiable at a, and*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Proof: From our hypotheses, there exist functions

$$D_a f\colon \mathrm{dom}(f) \to \mathbf{C}, \qquad D_{f(a)} g\colon \mathrm{dom}(g) \to \mathbf{C}$$

such that $D_a f$ is continuous at a, $D_{f(a)} g$ is continuous at $f(a)$ and

$$\begin{aligned} f(z) &= f(a) + (z-a) D_a f(z) \text{ for all } z \in \mathrm{dom}(f) && (10.14) \\ g(z) &= g\left(f(a)\right) + \left(z - f(a)\right) D_{f(a)} g(z) \text{ for all } z \in \mathrm{dom}(g). && (10.15) \end{aligned}$$

If $z \in \mathrm{dom}(g \circ f)$, then $f(z) \in \mathrm{dom}(g)$, so we can replace z in (10.15) by $f(z)$ to get

$$g\left(f(z)\right) = g\left(f(a)\right) + \left(f(z) - f(a)\right) D_{f(a)} g\left(f(z)\right) \text{ for all } z \in \mathrm{dom}(g \circ f).$$

Using (10.14) to rewrite $f(z) - f(a)$, we get

$$(g \circ f)(z) = (g \circ f)(a) + (z-a) D_a f(z) (D_{f(a)} g \circ f)(z) \text{ for all } z \in \mathrm{dom}(g \circ f).$$

Hence we have

$$D_a(g \circ f) = D_a f \cdot \Big((D_{f(a)} g) \circ f \Big)$$

and $D_a(g \circ f)$ is continuous at a. Hence $g \circ f$ is differentiable at a and

$$\begin{aligned} (g \circ f)'(a) &= D_a(g \circ f)(a) = D_a f(a) D_{f(a)} g\left(f(a)\right) \\ &= f'(a) \cdot g'\left(f(a)\right). \ \| \end{aligned}$$

10.16 Theorem (Reciprocal rule.) *Let f be a complex function, and let $a \in \mathrm{dom}(f)$. If f is differentiable at a and $f(a) \neq 0$, then $\dfrac{1}{f}$ is differentiable at a and $\left(\dfrac{1}{f}\right)'(a) = \dfrac{-f'(a)}{\left(f(a)\right)^2}$.*

Proof: If $v(z) = \dfrac{1}{z}$ for all $z \in \mathbf{C} \setminus \{0\}$, we saw above that v is differentiable and $v'(z) = -\dfrac{1}{z^2}$. Let f be a complex function, and let $a \in \mathbf{C}$. Suppose f is differentiable at a, and $f(a) \neq 0$. Then $(v \circ f)(z) = \dfrac{1}{f(z)}$. By the chain rule $v \circ f$ is differentiable at a, and

$$(v \circ f)'(a) = v'\left(f(a)\right) \cdot f'(a) = -\frac{1}{f(a)^2} f'(a). \ \|$$

10.17 Exercise (Product rule.) Let f, g be complex functions. Suppose f and g are both differentiable at a, and that a is a limit point of $\text{dom}(f) \cap \text{dom}(g)$. Show that fg is differentiable at a, and that

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

10.18 Exercise (Power rule.) Let f be a complex function, and suppose that f is differentiable at $a \in \mathbf{C}$. Show that f^n is differentiable at a for all $n \in \mathbf{Z}_{\geq 1}$ and

$$(f^n)'(a) = n\,(f(a))^{n-1}\, f'(a).$$

(Use induction.)

10.19 Exercise (Power rule.) Let f be a complex function. suppose that f is differentiable at $a \in \mathbf{C}$, and $f(a) \neq 0$. Show that f^n is differentiable at a for all $n \in \mathbf{Z}^-$, and that

$$(f^n)'(a) = n\,(f(a))^{n-1}\, f'(a).$$

for all $n \in \mathbf{Z}^-$.

10.20 Exercise (Quotient rule.) Let f, g be complex functions and let $a \in \mathbf{C}$. Suppose f and g are differentiable at a and $g(a) \neq 0$, and a is a limit point of $\text{dom}\left(\dfrac{f}{g}\right)$. Show that $\dfrac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g(a)^2}.$$

10.2 Differentiable Functions on R

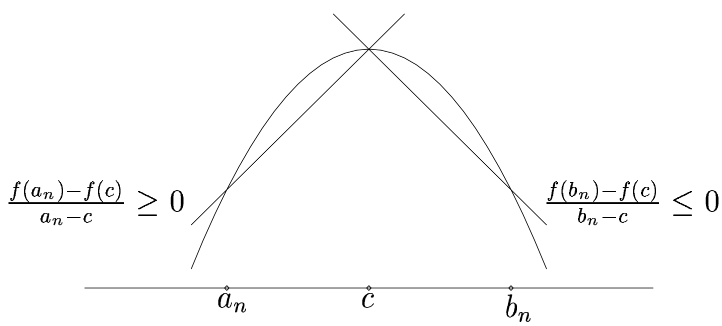
10.21 Warning. By the definition of differentiablity given in Math 111, the domain of a function was required to contain some interval $(a-\varepsilon, a+\varepsilon)$ in order for the function be differentiable at a. In definition 10.1 this condition has been replaced by requiring a to be a limit point of the domain of the function. Now a function whose domain is a closed interval $[a, b]$ may be differentiable at a and/or b.

10.22 Definition (Critical point.) Let f be a complex function, and let $a \in \mathbf{C}$. If f is differentiable at a and $f'(a) = 0$, we call a a *critical point* for f.

10.23 Theorem (Critical Point Theorem.) *Let $f\colon \text{dom}(f) \to \mathbf{R}$ be a function. Suppose f has a maximum at some point $c \in \text{dom}(f)$, and that $\text{dom}(f)$ contains an interval $(c-\varepsilon, c+\varepsilon)$ where $\varepsilon \in \mathbf{R}^+$. If f is differentiable at c, then $f'(c) = 0$. The theorem also holds if we replace "maximum" by "minimum."*

Proof: Suppose f has a maximum at c,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$



Define two sequences $\{a_n\}$, $\{b_n\}$ in $(c-\varepsilon, c+\varepsilon)$ by

$$\begin{aligned} a_n &= c - \frac{\varepsilon}{(n+2)} \text{ for all } n \in \mathbf{N} \\ b_n &= c + \frac{\varepsilon}{(n+2)} \text{ for all } n \in \mathbf{N}. \end{aligned}$$

Clearly $\{a_n\} \to c$ and $\{b_n\} \to c$, and $f(a_n) \leq f(c)$ and $f(b_n) \leq f(c)$ for all $n \in \mathbf{N}$. We have

$$\frac{f(a_n) - f(c)}{a_n - c} = \frac{f(a_n) - f(c)}{-\left(\frac{\varepsilon}{n+2}\right)} \geq 0.$$

By the inequality theorem,

$$f'(c) = \lim \left\{ \frac{f(a_n) - f(c)}{a_n - c} \right\} \geq 0.$$

Also,

$$\frac{f(b_n) - f(c)}{b_n - c} = \frac{f(b_n) - f(c)}{\left(\frac{\varepsilon}{n+2}\right)} \leq 0,$$

so

$$f'(c) = \lim \left\{ \frac{f(b_n) - f(c)}{b_n - c} \right\} \leq 0.$$

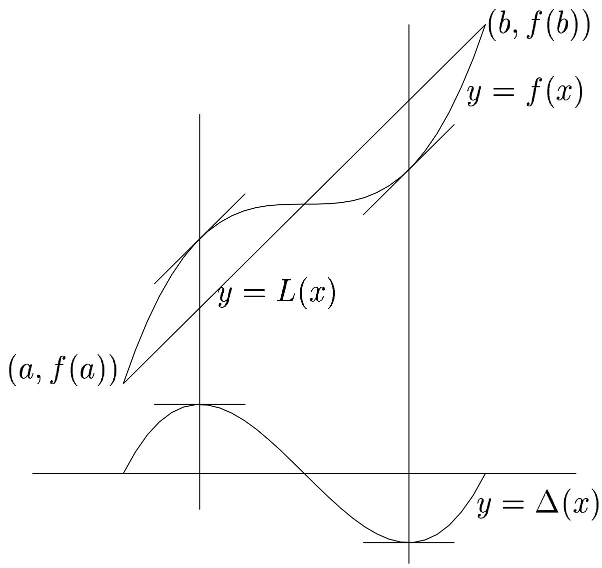
Since $0 \leq f'(c) \leq 0$, we conclude that $f'(c) = 0$. The proof for minimum points is left to you. |||

10.24 Theorem (Rolle's Theorem.) *Let $a, b \in \mathbf{R}$ with $a < b$ and let $f\colon [a, b] \to \mathbf{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = f(b)$. Then there is a number $c \in (a, b)$ such that $f'(c) = 0$.*

Proof: We know from the extreme value theorem that f has a maximum at some point $p \in [a, b]$. If $p \in (a, b)$, then the critical point theorem says $f'(p) = 0$, and we are finished. Suppose $p \in \{a, b\}$. We know there is a point $q \in [a, b]$ such that f has a minimum at q. If $q \in (a, b)$ we get $f'(q) = 0$ by the critical point theorem, so suppose $q \in \{a, b\}$. Then since $f(a) = f(b)$ and $p \in \{a, b\}, q \in \{a, b\}$, we have $f(p) = f(q)$, and it follows that f is a constant function on $[a, b]$, and in this case $f'(c) = 0$ for all $c \in (a, b)$. |||

10.25 Theorem (Mean Value Theorem.) *Let $a, b \in \mathbf{R}$ with $a < b$, and let f be a function from $[a, b]$ to $\mathbf{R}$ such that f is continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



This theorem says that the tangent to the graph of f at some point $(c, f(c))$ is parallel to the chord joining $(a, f(a))$ to $(b, f(b))$.
Proof: Let

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \text{ for all } x \in \mathbf{R},$$

so the equation of the line joining $(a, f(a))$ to $(b, f(b))$ is $y = L(x)$, and

$$L'(x) = \frac{f(b) - f(a)}{b - a} \text{ for all } x \in \mathbf{R}.$$

Let

$$\Delta(x) = f(x) - L(x) \text{ for all } x \in [a, b].$$

Then

$$\begin{aligned} \Delta(a) &= f(a) - L(a) = f(a) - f(a) = 0, \\ \Delta(b) &= f(b) - L(b) = f(b) - f(b) = 0, \end{aligned}$$

and Δ is continuous on $[a, b]$ and differentiable on (a, b). By Rolle's theorem, there is some $c \in (a, b)$ such that $\Delta'(c) = 0$; i.e., $f'(c) - L'(c) = 0$; i.e.,

$$f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a}. \ \|$$

10.26 Remark. The mean value theorem does not hold for complex valued functions. Let

$$F(t) = (1 + it)^4 \text{ for all } t \in [-1, 1].$$

Then

$$F(\pm 1) = (1 \pm i)^4 = (\pm 2i)^2 = -4,$$

so

$$\frac{F(1) - F(-1)}{1 - (-1)} = 0.$$

But

$$F'(t) = 4i(1 + it)^3,$$

so $F'(t) = 0 \iff t = -i$, and there is no point in $t \in (-1, 1)$ with $F'(t) = 0$.

10.27 Definition (Interior point.) Let J be an interval in $\mathbf{R}$. A number $a \in J$ is an *interior point* of J if and only if a is not an end point of J. The set of all interior points of J is called the *interior* of J and is denoted by $\text{int}(J)$.

10.28 Examples. If $a < b$, then

$$\begin{aligned} \mathrm{int}([a,b)) = \mathrm{int}([a,b]) = \mathrm{int}((a,b)) &= (a,b), \\ \mathrm{int}([a,\infty)) = \mathrm{int}((a,\infty)) &= (a,\infty). \end{aligned}$$

If J is an interval, and s,t are points in J with $s < t$, then every point in (s,t) is in the interior of J.

10.29 Theorem. *Let J be an interval in* **R***, and let $f\colon J \to \mathbf{R}$ be a continuous function on J. Then:*

a) If $f'(x) \geq 0$ for all $x \in \mathrm{int}(J)$, then f is increasing on J.

b) If $f'(x) > 0$ for all $x \in \mathrm{int}(J)$, then f is strictly increasing on J.

c) If $f'(x) \leq 0$ for all $x \in \mathrm{int}(J)$, then f is decreasing on J.

d) If $f'(x) < 0$ for all $x \in \mathrm{int}(J)$, then f is strictly decreasing on J.

e) If $f'(x) = 0$ for all $x \in \mathrm{int}(J)$, then f is constant on J.

Proof: All five statements have similar proofs. I'll prove only part a).

Suppose $f'(x) \geq 0$ for all $x \in \mathrm{int}(J)$. Then for all $s,t \in J$ with $s < t$ we have f is continuous on $[s,t]$ and differentiable on (s,t), so by the mean value theorem

$$\begin{aligned} s < t &\implies \frac{f(t)-f(s)}{t-s} = f'(c) \text{ for some } c \in (s,t) \subset \mathrm{int}(J) \\ &\implies \frac{f(t)-f(s)}{t-s} \geq 0 \text{ and } t-s > 0 \\ &\implies f(t)-f(s) \geq 0 \\ &\implies f(t) \geq f(s). \end{aligned}$$

Hence, f is increasing on J.

10.30 Exercise. Prove part e) of the previous theorem; i.e., show that if J is an interval in **R** and $f\colon J \to \mathbf{R}$ is continuous and satisfies $f'(t) = 0$ for all $t \in \mathrm{int}(J)$, then f is constant on J. [It is sufficient to show that $f(s) = f(t)$ for all $s,t \in J$.]

10.31 Exercise. For each assertion below, either prove that the assertion is true for all functions f, or give a function f for which the assertion is false. (A proof may consist of quoting a theorem.)

a) If f is differentiable on $(-1,1)$ and f is strictly increasing on $(-1,1)$, then $f'(t) > 0$ for all $t \in (-1,1)$.

b) If f is differentiable on $[-1,1]$, and f has a maximum at $t_0 \in [-1,1]$, then $f'(t_0) = 0$.

c) If f is continuous on $[-1,1]$ and f is differentiable on $(-1,1)$, and $f'(t) > 0$ for all $t \in (-1,1)$, then f is strictly increasing on $[-1,1]$.

10.32 Theorem (Restriction theorem) *Let S be a subset of* $\mathbf{C}$*, let $f : S \to \mathbf{C}$, and let $a \in S$ be a point such that f is differentiable at a. Let T be a subset of S containing a, and let $f|_T : T \to \mathbf{C}$ be the restriction of f to T, i.e.*

$$f|_T(z) = f(z) \text{ for all } z \in T.$$

If a is a limit point of T, then $f|_T$ is differentiable at a, and

$$f|_T'(a) = f'(a).$$

Proof: Let $\{z_n\}$ be any sequence in $T \setminus \{a\}$ such that $\{z_n\} \to a$. Then $\{z_n\}$ is a sequence in $S \setminus \{a\}$, and hence

$$\left\{\frac{f(z_n) - f(a)}{z_n - a}\right\} \to f'(a).$$

It follows that

$$\left\{\frac{f|_T(z_n) - f|_T(a)}{z_n - a}\right\} = \left\{\frac{f(z_n) - f(a)}{z_n - a}\right\} \to f'(a).$$

I've shown that

$$\lim_{z \to a} \frac{f|_T(z) - f|_T(a)}{z - a} = f'(a). \parallel$$

10.33 Definition (Path, line segment.) If $a, b \in \mathbf{C}$, then the *path joining a to b* is the function $\lambda_{ab}\colon [0,1] \to \mathbf{C}$

$$\lambda_{ab}\colon t \mapsto a + t(b-a) \text{ for all } t \in [0,1]$$

and the set

$$\Lambda_{ab} = \lambda_{ab}([0,1]) = \{a + t(b-a)\colon 0 \le t \le 1\}$$

is called the *line segment* joining a to b.

10.34 Example. We showed in example 10.9 that the function conj : $z \mapsto z^*$ is a nowhere differentiable function on **C**. I will show that for all a, b in **C** with $a \neq b$, the restriction $\text{conj}|_{\Lambda_{ab}}$ of conj to the line segment Λ_{ab} is differentiable, and

$$\text{conj}|_{\Lambda_{ab}}{}'(z) = \frac{b^* - a^*}{b - a} \text{ for all } z \in \Lambda_{ab}.$$

Note that all points of Λ_{ab} are limit points of Λ_{ab}. If $z \in \Lambda_{ab}$, then for some real number t

$$z = a + t(b - a) \tag{10.35}$$

and

$$z^* = a^* + t(b^* - a^*). \tag{10.36}$$

If we solve equation (10.35) for t we get

$$t = \frac{z - a}{b - a}.$$

By using this value for t in equation (10.36) we get

$$z^* = a^* + \frac{b^* - a^*}{b - a}(z - a) \text{ for all } z \in \Lambda_{ab}.$$

Let $H_{ab} : \mathbf{C} \to \mathbf{C}$ be defined by

$$H_{ab}(z) = a^* + \frac{b^* - a^*}{b - a}(z - a) \text{ for all } z \in \mathbf{C}.$$

Then H_{ab} is differentiable, and $H'(z) = \dfrac{b^* - a^*}{b - a}$ for all $z \in \mathbf{C}$. We have

$$H_{ab}|_{\Lambda_{ab}} = \text{conj}|_{\Lambda_{ab}},$$

so by the restriction theorem

$$\text{conj}|_{\Lambda_{ab}}{}'(z) = H|_{\Lambda_{ab}}{}'(z) = \frac{b^* - a^*}{b - a} \text{ for all } z \in \Lambda_{ab}.$$

10.37 Exercise. Let $C(0,1)$ denote the unit circle in **C**. Show that $\text{conj}|_{C(0,1)}$ is differentiable, and that

$$\text{conj}_{C(0,1)}{}'(z) = -(z^*)^2 \text{ for all } z \in C(0,1).$$

In general, the real and imaginary parts of a differentiable function are not differentiable.

10.38 Example. If $f(z) = z$ for all $z \in \mathbf{C}$, then f is differentiable and $f'(z) = 1$. However, Ref is nowhere differentiable. In fact, if $a \in \mathbf{C}$, $\dfrac{\text{Re}(z) - \text{Re}(z)}{z - a}$ has no limit at a. To see this, let $a_n = a + \dfrac{1}{n}$, $b_n = a + \dfrac{i}{n}$ for all $n \in \mathbf{Z}_{\geq 1}$. Then $\{a_n\} \to a$, $\{b_n\} \to a$, and

$$\frac{\text{Re}(a_n) - \text{Re}(a)}{a_n - a} = \frac{\text{Re}(a + \frac{1}{n}) - \text{Re}(a)}{a + \frac{1}{n} - a} = 1$$

and

$$\frac{\text{Re}(b_n) - \text{Re}(a)}{b_n - a} = \frac{\text{Re}(a) - \text{Re}(a)}{a + \frac{i}{n} - a} = 0.$$

Hence, the sequences $\left\{\dfrac{\text{Re}(a_n) - \text{Re}(a)}{a_n - a}\right\}_{n\geq 1}$ and $\left\{\dfrac{\text{Re}(b_n) - \text{Re}(a)}{b_n - a}\right\}_{n\geq 1}$ have different limits, so $\lim\limits_{z\to a} \dfrac{\text{Re}(z) - \text{Re}(a)}{z - a}$ does not exist.

However, we do have the following theorem.

10.39 Theorem. *Let J be an interval in $\mathbf{R}$ and let $f \colon J \to \mathbf{C}$ be a function differentiable at a point $a \in J$. Write $f(t) = u(t) + iv(t)$ where u, v are real valued. Then u and v are differentiable at a, and $f'(a) = u'(a) + iv'(a)$.*

Proof: Since f is differentiable at a there is a function $D_a f$ on J such that $D_a f$ is continuous at a and

$$f(t) = f(a) + (t - a)D_a f(t) \text{ for all } t \in J.$$

If $r \in \mathbf{R}$ and $c \in \mathbf{C}$, then $\text{Re}(rc) = r\text{Re}(c)$ and $\text{Im}(rc) = r\text{Im}(c)$, so

$$(\text{Re}(f))(t) = (\text{Re}(f))(a) + (t - a)(\text{Re}(D_a f))(t) \text{ for all } t \in J \tag{10.40}$$

and

$$(\text{Im}(f))(t) = (\text{Im}(f))(a) + (t - a)(\text{Im}(D_a f))(t) \text{ for all } t \in J. \tag{10.41}$$

Since $D_a f$ is continuous at a, $\text{Re}(D_a f)$ and $\text{Im}(D_a f)$ are continuous at a, so equations (10.40) and (10.41) show that Ref and Imf are differentiable and

$$\begin{aligned} (\text{Re}f)'(a) &= (\text{Re}(D_a f(a)) = \text{Re}(f'(a)) \\ (\text{Im}f)'(a) &= (\text{Im}\,(D_a f(a))) = \text{Im}(f'(a)). \ \| \end{aligned}$$

10.42 Example. Let $a \in \mathbf{R}$, and let $f(t) = (2t + ia)^3$ for all $t \in \mathbf{R}$. Then f is differentiable and (by the chain rule),

$$\begin{aligned} f'(t) &= 3(2t + ia)^2 \cdot 2 \\ &= 6[(4t^2 - a^2) + 4iat] \\ &= (24t^2 - 6a^2) + 24iat. \end{aligned}$$

We have by direct calculation,

$$\begin{aligned} f(t) &= 8t^3 + 12iat^2 - 6ta^2 - ia^3 \\ &= (8t^3 - 6ta^2) + (12at^2 - a^3)i, \end{aligned}$$

so

$$f'(t) = (24t^2 - 6a^2) + (24at)i.$$

(This example just illustrates that the theorem is true in a special case.)

10.43 Theorem. *Let f be a complex function and let $a, b \in \mathbf{C}$, and suppose* dom(f) *contains the line segment Λ_{ab}, and that $f'(z) = 0$ for all $z \in \Lambda_{ab}$. Then f is constant on Λ_{ab}; i.e., $f(z) = f(a)$ for all $z \in \Lambda_{ab}$.*

Proof: Define a function $F \colon [0,1] \to \mathbf{C}$ by

$$F(t) = f\left(\lambda_{ab}(t)\right) = f\left(a + t(b - a)\right).$$

By the chain rule, F is differentiable on $[0,1]$ and $F'(t) = f'\left(a + t(b-a)\right) \cdot (b-a)$. Since $f'(z) = 0$ for all $z \in \Lambda_{ab}([0,1])$, we have $F'(t) = 0$ for all $t \in [0,1]$. Hence

$$(\mathrm{Re}(F))'(t) = 0 \text{ and } (\mathrm{Im}(F))'(t) = 0 \text{ for all } t \in [0,1]$$

and hence

$$\mathrm{Re}(F) \text{ and } \mathrm{Im}(F) \text{ are constant on } [0,1].$$

If $\mathrm{Re}(F) = p$ and $\mathrm{Im}(F) = q$, then $F(t) = p + iq$ for all $t \in [0,1]$. $\|$

10.44 Exercise. Let $D(a, \varepsilon)$ be a disc in $\mathbf{C}$.

a) Show that if $b \in D(a, \varepsilon)$ then the segment Λ_{ab} is a subset of $D(a, \varepsilon)$.

b) Let $f \colon D(a, \varepsilon) \to \mathbf{C}$ be a function such that $f'(z) = 0$ for all $z \in D(a, \varepsilon)$. Show that f is constant on $D(a, \varepsilon)$.

10.3 Trigonometric Functions

10.45 Example. Suppose that there are real valued functions S, C on $\mathbf{R}$ such that

$$\begin{aligned} S' &= C, & S(0) &= 0, \\ C' &= -S, & C(0) &= 1. \end{aligned}$$

You have seen such functions in your previous calculus course. Let $H = S^2 + C^2$. Then

$$H' = 2SS' + 2CC' = 2SC - 2CS = 0.$$

Hence, H is constant on $\mathbf{R}$, and since $H(0) = S^2(0) + C^2(0) = 0 + 1$, we have

$$S^2 + C^2 = \tilde{1} \text{ on } \mathbf{R}.$$

In particular,

$$|S(t)| \leq 1 \text{ and } |C(t)| \leq 1 \text{ for all } t \in \mathbf{R}.$$

Let $K(t) = (S(t) + S(-t))^2 + (C(t) - C(-t))^2$. By the power rule and chain rule,

$$\begin{aligned} &K'(t) \\ &= 2(S(t) + S(-t))(S'(t) - S'(-t)) + 2(C(t) - C(-t))(C'(t) + C'(-t)) \\ &= 2(S(t) + S(-t))(C(t) - C(-t)) + 2(C(t) - C(-t))(-S(t) - S(-t)) \\ &= 0. \end{aligned}$$

Hence K is constant and since $K(0) = 0$, we conclude that $K(t) = 0$ for all t. Since a sum of squares in $\mathbf{R}$ is zero only when each summand is zero, we conclude that

$$\begin{aligned} S(-t) &= -S(t) \text{ for all } t \in \mathbf{R}, \\ C(-t) &= C(t) \text{ for all } t \in \mathbf{R}. \end{aligned}$$

Let

$$F_0(t) = -C(t) + 1 \text{ for all } t \in \mathbf{R}.$$

Then $F_0(t) \geq 0$ for all $t \in \mathbf{R}$ and $F_0(0) = 0$. I will now construct a sequence $\{F_n\}$ of functions on $\mathbf{R}$ such that $F_n(0) = 0$ for all $n \in \mathbf{N}$, and $F'_{n+1}(t) = F_n(t)$

for all $t \in \mathbf{R}$. I have

$$\begin{aligned} F_1(t) &= -S(t) + t, \\ F_2(t) &= C(t) + \frac{t^2}{2!} - 1, \\ F_3(t) &= S(t) + \frac{t^3}{3!} - \frac{t}{1!}, \\ F_4(t) &= -C(t) + \frac{t^4}{4!} - \frac{t^2}{2!} + 1, \\ F_5(t) &= -S(t) + \frac{t^5}{5!} - \frac{t^3}{3!} + t. \end{aligned}$$

It should be clear how this pattern continues. Since $F_1'(t) = F_0(t) \geq 0$, F_1 is increasing on $[0, \infty)$ and since $F_1(0) = 0$, $F_1(t) \geq 0$ for $t \in [0, \infty)$. Since $F_2'(t) = F_1(t) \geq 0$ on $[0, \infty)$, F_2 is increasing on $[0, \infty)$ and since $F_2(0) = 0$, $F_2(t) \geq 0$ for $t \in [0, \infty)$.

This argument continues (I'll omit the inductions), and I conclude that $F_n(t) \geq 0$ for all $t \in [0, \infty)$ and all $n \in \mathbf{N}$. Now

$$\begin{aligned} F_0(t) \geq 0 \text{ and } F_2(t) \geq 0 &\implies \frac{-t^2}{2!} \leq C(t) - 1 \leq 0, \\ F_1(t) \geq 0 \text{ and } F_3(t) \geq 0 &\implies \frac{-t^3}{3!} \leq S(t) - t \leq 0, \\ F_2(t) \geq 0 \text{ and } F_4(t) \geq 0 &\implies 0 \leq C(t) - 1 + \frac{t^2}{2!} \leq \frac{t^4}{4!}, \\ F_3(t) \geq 0 \text{ and } F_5(t) \geq 0 &\implies 0 \leq S(t) - t + \frac{t^3}{3!} \leq \frac{t^5}{5!}, \\ F_4(t) \geq 0 \text{ and } F_6(t) \geq 0 &\implies \frac{-t^6}{6!} \leq C(t) - 1 + \frac{t^2}{2!} - \frac{t^4}{4!} \leq 0. \end{aligned}$$

For each $n \in \mathbf{N}$, $t \in \mathbf{C}$, define

$$\begin{aligned} c_n(t) &= \frac{(-1)^n t^{2n}}{(2n)!}, \\ s_n(t) &= \frac{(-1)^n t^{2n+1}}{(2n+1)!}, \\ C_n(t) &= \sum_{j=0}^{n} c_j(t) = \sum_{j=0}^{n} \frac{(-1)^j t^{2j}}{(2j)!}, \\ S_n(t) &= \sum_{j=0}^{n} s_j(t) = \sum_{j=0}^{n} \frac{(-1) t^{2j+1}}{(2j+1)!}. \end{aligned}$$

The equations above suggest that for all $n \in \mathbf{N}$, $t \in [0, \infty)$,

$$|C(t) - C_n(t)| \leq |c_{n+1}(t)| \tag{10.46}$$

and

$$|S(t) - S_n(t)| \leq |s_{n+1}(t)| \tag{10.47}$$

I will not write down the induction proof for this because I believe that it is clear from the examples how the proof goes, but the notation becomes complicated.

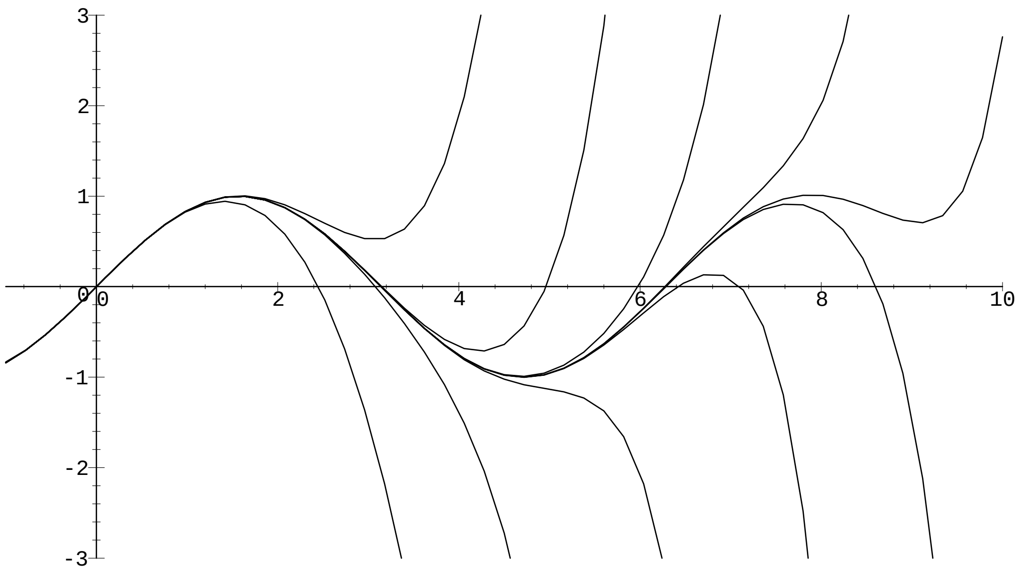
Since $C(t) = C(-t)$, $C_n(t) = C_n(-t)$ and $c_n(t) = c_n(-t)$, the relation (10.46) actually holds for all $t \in \mathbf{R}$ (not just for $t \in [0, \infty)$) and similarly relation (10.47) holds for all $t \in \mathbf{R}$. From (10.46) and (10.47), we see that if $\{c_n(t)\}$ is a null sequence, then the sequence $\{C_n(t)\}$ converges to $C(t)$, and if $\{s_n(t)\}$ is a null sequence, then $\{S_n(t)\}$ converges to $S(t)$.

We will show later that both sequences $\{C_n(z)\}$ and $\{S_n(z)\}$ converge for all *complex* numbers z, and we will define

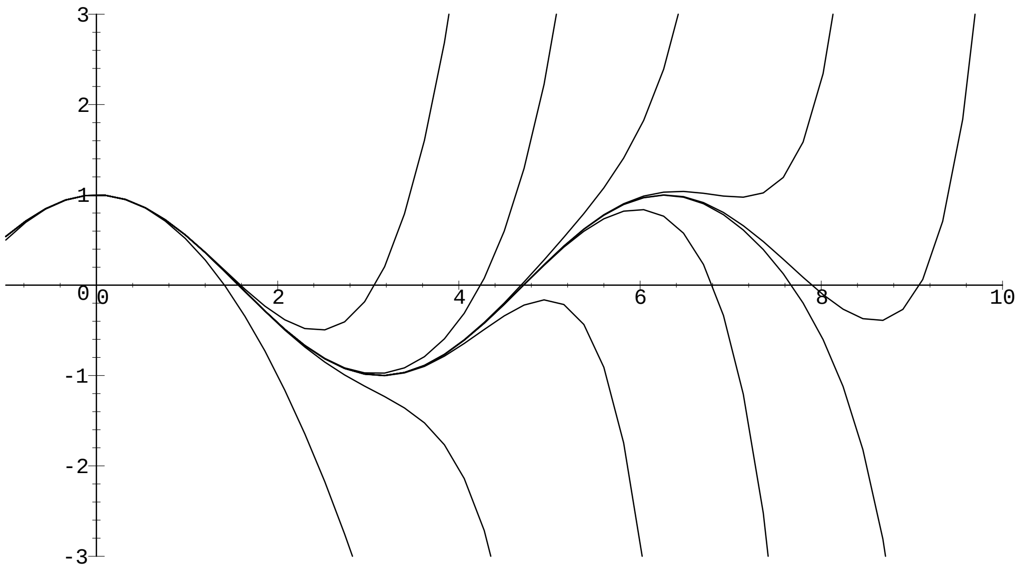
$$\cos(z) = \lim\{C_n(z)\} = \lim\left\{\sum_{j=0}^{n} \frac{(-1)^j z^{2j}}{(2j)!}\right\} \tag{10.48}$$

$$\sin(z) = \lim\{S_n(z)\} = \lim\left\{\sum_{j=0}^{n} \frac{(-1)^j z^{2j+1}}{(2j+1)!}\right\} \tag{10.49}$$

for all $z \in \mathbf{C}$. The discussion above is supposed to convince you that for real z this definition agrees with whatever definition of sine and cosine you are familiar with. The figures show graphs of C_n and S_n for small n.



Graphs of the polynomials S_n for $1 \leq n \leq 10$

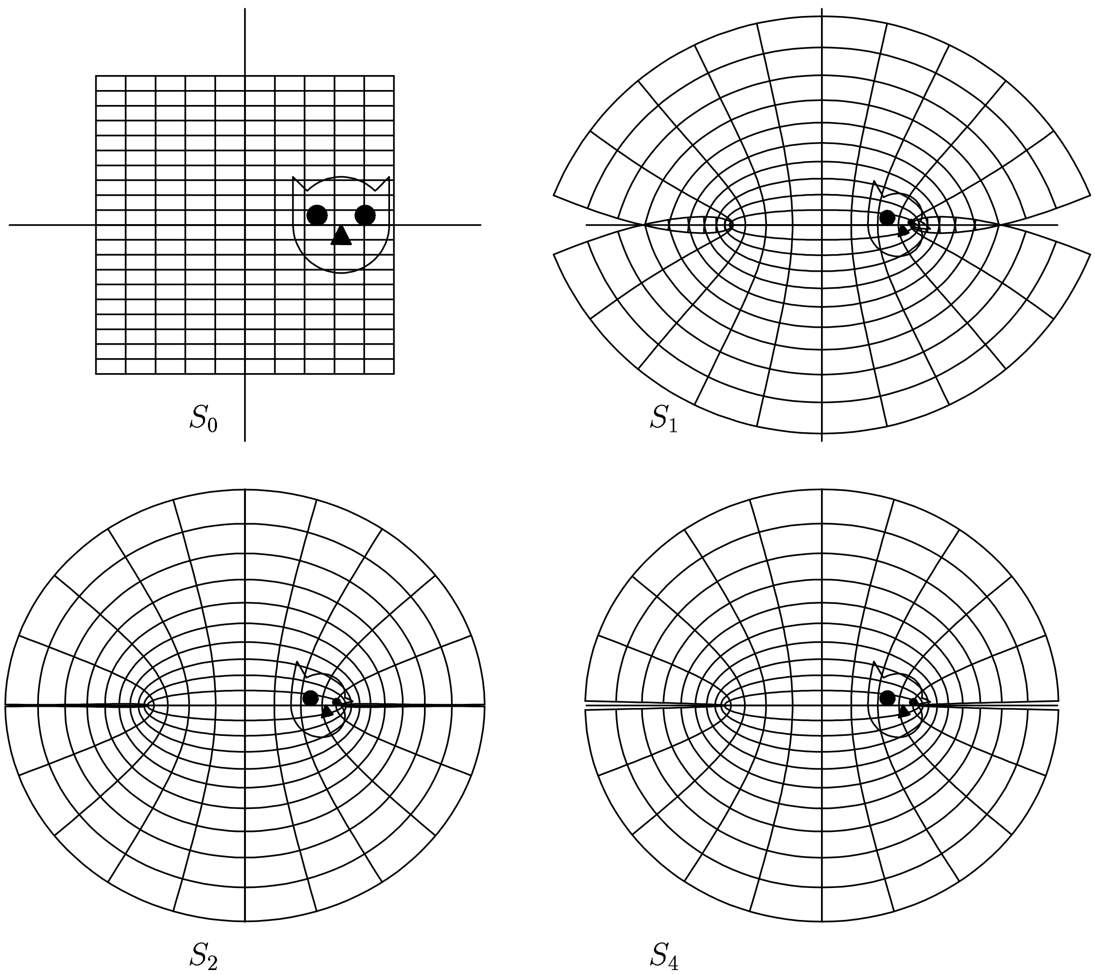


Graphs of the polynomials C_n for $1 \leq n \leq 10$

10.50 Exercise. Show that $\{c_n(t)\}$ and $\{s_n(t)\}$ are null sequences for all complex t with $|t| \leq 1$.

10.51 Exercise. a) Using calculator arithmetic, calculate the limits of $\left\{C_n\left(\frac{1}{10}\right)\right\}$ and $\left\{S_n\left(\frac{1}{10}\right)\right\}$ accurate to 8 decimals. Compare your results with your calculator's value of $\sin\left(\frac{1}{10}\right)$ and $\cos\left(\frac{1}{10}\right)$. [Be sure to use radian mode.]

b) Calculate $\cos(i)$ to 3 or 4 decimals accuracy. Note that $\cos(i)$ is real.



Polynomial Approximations to sine Function
$-1.55 \leq x \leq 1.55, \quad -1.55 \leq y \leq 1.55$

The figure shows graphical representations for S_0, S_1, S_2, and S_4. Note that S_0 is the identity function.

10.52 Entertainment. Show that for all $a, x \in \mathbf{R}$

$$C(a+x) = C(a)C(x) - S(a)S(x)$$

and

$$S(a+x) = S(a)C(x) + C(a)S(x).$$

Use a trick similar to the trick used to show that $S(-x) = -S(x)$ and $C(-x) = C(x)$.

10.53 Entertainment. By using the definitions (10.48) and (10.49), show that

a) For all $a \in \mathbf{R}$, $\cos(ia)$ is real, and $\cos(ia) \geq 1$.

b) For all $a \in \mathbf{R}$, $\sin(ia)$ is pure imaginary, and $\sin(ia) = 0$ if and only if $a = 0$.

c) Assuming that the identity

$$\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

is valid for all complex numbers z and w, show that if $a \in \mathbf{R} \setminus \{0\}$ then sin maps the horizontal line $y = a$ to the ellipse having the equation

$$\frac{x^2}{|\cos(ia)|^2} + \frac{y^2}{|\sin(ia)|^2} = 1.$$

d) Describe where sin maps vertical lines. (Assume that the identity $\sin^2(z) + \cos^2(z) = 1$ holds for all $z \in \mathbf{C}$.)

10.54 Note. Rolle's theorem is named after Michel Rolle (1652–1719). An English translation of Rolle's original statement and proof can be found in [46, pages 253–260]. It takes a considerable effort to see any relation between what Rolle says, and what our form of his theorem says.

The series representations for sine and cosine (10.48) and (10.49) are usually credited to Newton, who discovered them some time around 1669. However, they were known in India centuries before this. Several sixteenth century Indian writers quote the formulas and attribute them to Madhava of Sangamagramma (c. 1340-1425)[30, p 294].

The method used for finding the series for sine and cosine appears in the 1941 book *What is Mathematics"* by Courant and Robbins[17, page 474]. I expect that the method was well known at that time.