Chapter 8

Continuity

8.1 Compositions with Sequences

8.1 Definition (Composition) Let $a: n \mapsto a(n)$ be a complex sequence. Let $g: S \to \mathbf{C}$ be a function such that $\text{dom}(g) = S \subset \mathbf{C}$, and $a(n) \in S$ for all $n \in \mathbf{N}$. Then the *composition* $g \circ a$ is the sequence such that $(g \circ a)(n) = g\,(a(n))$ for all $n \in \mathbf{N}$. If a is a sequence, I will often write a_n instead of $a(n)$. Then

$$a = \{a_n\} \implies g \circ a = \{g(a_n)\}.$$

8.2 Examples. If

$$f = \left\{\frac{1}{2^n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{4}, \cdots\right\}$$

and $g(z) = \dfrac{1}{1+z}$ for all $z \in \mathbf{C} \setminus \{-1\}$, then

$$g \circ f = \left\{\frac{1}{1 + \frac{1}{2^n}}\right\} = \left\{\frac{2^n}{2^n + 1}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{4}{5}, \cdots\right\}.$$

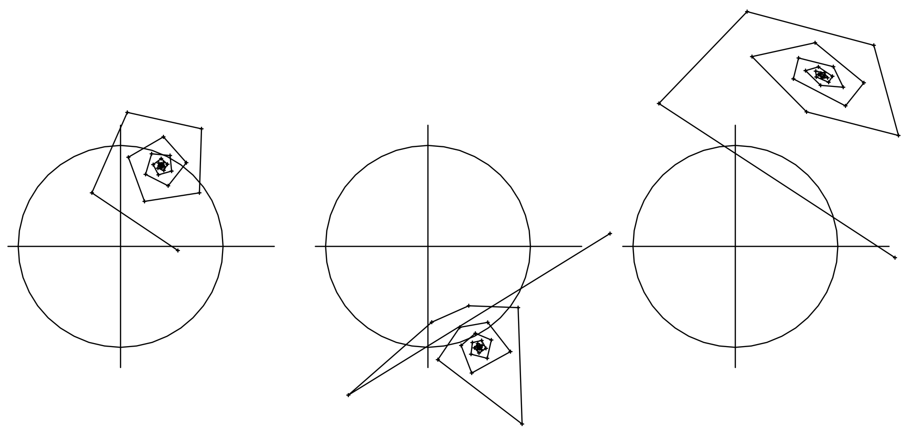
Figure a) below shows representations of $x, v \circ x$ and $K \circ x$ where

$$\begin{aligned} x(n) &= x_n = \frac{2}{5} + \frac{4}{5}i + \left(\frac{4 - 21i}{25}\right)^{n+1} \text{ for all } n \in \mathbf{N}, \\ v(z) &= \frac{1}{z} \text{ for all } z \in \mathbf{C} \setminus \{0\}, \\ K(z) &= z + \frac{z}{|z|} \text{ for all } z \in \mathbf{C} \setminus \{0\}. \end{aligned} \tag{8.3}$$

I leave it to you to check that $\{x_n\} \to \dfrac{2}{5} + \dfrac{4}{5}i$, and $\{v(x_n)\} \to v\left(\dfrac{2}{5} + \dfrac{4}{5}i\right)$, and $\{K(x_n)\} \to K\left(\dfrac{2}{5} + \dfrac{4}{5}i\right)$. Figure b) shows representations for a and $K \circ a$ where

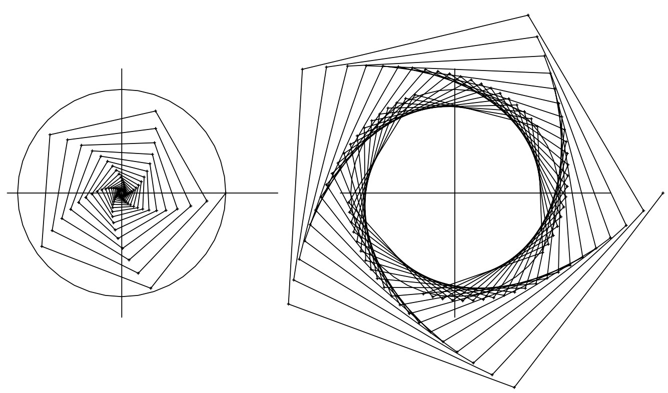
$$a(n) = a_n = \left(\frac{7 - 23i}{25}\right)^n,$$

and K is defined as in (8.3). Here it is easy to check that $\{a_n\} \to 0$. From the figure, $\{K(a_n)\}$ doesn't appear to converge.



$\{x_n\}$ $\{v(x_n)\}$ $\{K(x_n)\}$

figure a) Sequence and compositions



$\{a_n\}$ $\{K(a_n)\}$

figure b) Sequence and composition

8.4 Exercise. Let α be a non-zero complex number with $0 < |\alpha| < 1$. Let

$$f_\alpha(n) = \alpha^n + \frac{\alpha^n}{|\alpha^n|} \text{ for all } n \in \mathbf{N}.$$

Under what conditions on α does f_α converge? What does it converge to? (Your answer should show that the sequence $\{K(a_n)\}$ from the previous example does not converge.)

8.5 Definition (Complex function.) By a *complex function* I will mean a function whose domain is a subset of $\mathbf{C}$, and whose codomain is $\mathbf{C}$. I will consider functions from $\mathbf{R}$ to $\mathbf{R}$ to be complex functions by identifying a function $f\colon S \to \mathbf{R}$ with a function $f\colon S \to \mathbf{C}$ in the expected manner.

8.2 Continuity

8.6 Definition (Continuous) Let f be a complex function and let $p \in \text{dom}(f)$. We say *f is continuous at p* if

$$\text{for every sequence } x \text{ in } \text{dom}(f) \quad (x \to p \implies f \circ x \to f(p))\,;$$

i.e., if

$$\text{for every sequence } \{x_n\} \text{ in } \text{dom}(f) \quad (\{x_n\} \to p \implies \{f(x_n)\} \to f(p))\,.$$

Let B be a subset of S. We say *f is continuous on B* if f is continuous at q for all $q \in B$. We say f is *continuous* if f is continuous on $\text{dom}(f)$; i.e., if f is continuous at every point at which it is defined.

8.7 Examples. If $f(z) = z$ for all $z \in \mathbf{C}$, then f is continuous. In this case $f \circ x = x$ for every sequence x so the condition for continuity at p is

$$x \to p \implies x \to p.$$

If $a \in \mathbf{C}$, then the constant function $\tilde{a}$ is continuous since for all $p \in \mathbf{C}$, and all complex sequences x,

$$x \to p \implies \tilde{a} \circ x = \tilde{a} \to a = \tilde{a}(p).$$

Notice that Re and Im (Real part and imaginary part) are functions from $\mathbf{C}$ to $\mathbf{R}$. In theorem 7.39 we showed if x is any complex sequence and $L \in \mathbf{C}$, then

$$x \to L \implies \mathrm{Re}(x) \to \mathrm{Re}(L)$$

and

$$x \to L \implies \mathrm{Im}(x) \to \mathrm{Im}(L).$$

Hence Re and Im are continuous functions on $\mathbf{C}$.

8.8 Theorem. *If* abs *and* conj *are functions from* $\mathbf{C}$ *to* $\mathbf{C}$ *defined by*

$$\begin{aligned} \mathrm{abs}(z) &= |z| \textit{ for all } z \in \mathbf{C} \\ \mathrm{conj}(z) &= z^* \textit{ for all } z \in \mathbf{C}, \end{aligned}$$

then abs *and* conj *are continuous.*

Proof: Let $a \in \mathbf{C}$ and let x be any sequence in $\mathbf{C}$ such that $\{x_n\} \to a$; i.e., $\{x_n - a\}$ is a null sequence. By the reverse triangle inequality,

$$|x_n - a| \geq |x_n| - |a|$$

and

$$|x_n - a| = |a - x_n| \geq |a| - |x_n|,$$

so we have

$$-|x_n - a| \leq |x_n| - |a| \leq |x_n - a|$$

and hence

$$||x_n| - |a|| \leq |x_n - a|.$$

It follows by the comparison theorem that $\{|x_n| - |a|\}$ is a null sequence; i.e., $\{|x_n|\} \to |a|$. Hence abs is continuous.

Since $|x_n^* - a^*| = |(x_n - a)^*| = |x_n - a|$, the comparison theorem shows that

$$\{x_n\} \to a \implies \{x_n^*\} \to a^*;$$

i.e., conj is continuous. |||

8.9 Example. If

$$f(z) = \begin{cases} z & \text{for } z \in \mathbf{C}\backslash\{0\}, \\ 1 & \text{for } z = 0, \end{cases}$$

then f is not continuous at 0, since

$$\left\{\frac{1}{n}\right\} \to 0$$

but

$$\left\{f\left(\frac{1}{n}\right)\right\} = \left\{\frac{1}{n}\right\} \to 0 \neq f(0).$$

Notice that to show that a function f is *not* continuous at a point a in its domain, it is sufficient to find *one* sequence $\{x_n\}$ in $\operatorname{dom}(f)$ such that $\{x_n\} \to a$ and either $\{f(x_n)\}$ converges to a limit different from $f(a)$ or $\{f(x_n)\}$ diverges.

8.10 Theorem (Sum and Product theorems.) *Let f, g be complex functions, and let $a \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$. If f and g are continuous at a, then $f+g$, $f-g$, and $f \cdot g$ are continuous at a.*

Proof: Let $\{x_n\}$ be a sequence in domain $(f+g)$ such that $\{x_n\} \to a$. Then $x_n \in \operatorname{dom}(f)$ for all n and $x_n \in \operatorname{dom}(g)$ for all n, and by continuity of f and g at a, it follows that

$$\{f(x_n)\} \to f(a) \text{ and } \{g(x_n)\} \to g(a).$$

By the sum theorem for sequences,

$$\{(f+g)(x_n)\} = \{f(x_n) + g(x_n)\} \to f(a) + g(a) = (f+g)(a).$$

Hence $f+g$ is continuous at a. The proofs of continuity for $f-g$ and $f \cdot g$ are similar.

8.11 Theorem (Quotient theorem.) *Let f, g be complex functions and let $a \in \operatorname{dom}\left(\frac{f}{g}\right)$. If f and g are continuous at a, then $\frac{f}{g}$ is continuous at a.*

8.12 Exercise. Prove the quotient theorem. Recall that

$$\operatorname{dom}\left(\frac{f}{g}\right) = (\operatorname{dom}(f) \cap \operatorname{dom}(g)) \backslash \{z \in \operatorname{dom}(g) \colon g(z) = 0\}.$$

8.13 Theorem (Continuity of roots.) *Let $p \in \mathbf{Z}_{\geq 1}$ and let $f_p(x) = x^{\frac{1}{p}}$ for all $x \in [0, \infty)$. Then f_p is continuous.*

Proof: First we show f_p is continuous at 0. Let $\{x_n\}$ be a sequence in $[0, \infty)$ such that $\{x_n\} \to 0$; i.e., such that $\{x_n\}$ is a null sequence. Then by the root theorem for null sequences (Theorem 7.19), $\left\{x_n^{\frac{1}{p}}\right\}$ is a null sequence; i.e., $\{f_p(x_n)\} = \left\{x_n^{\frac{1}{p}}\right\} \to 0 = f_p(0)$, so f_p is continuous at 0.

Next we show that f_p is continuous at 1. By the formula for a finite geometric series (3.72), we have for all $x \in [0, \infty)$

$$|x^p - 1| = \left|(x-1)\sum_{j=0}^{p-1} x^j\right| = |x-1|\sum_{j=0}^{p-1} x^j \geq |x-1|. \tag{8.14}$$

If we replace x by $y^{\frac{1}{p}}$ in (8.14), we get $|y-1| = |(y^{\frac{1}{p}})^p - 1| \geq |y^{\frac{1}{p}} - 1|$, i.e.,

$$|y^{\frac{1}{p}} - 1| \leq |y - 1| \text{ for all } y \in [0, \infty).$$

Let $\{y_n\}$ be a sequence in $[0, \infty)$. Then

$$|(y_n)^{\frac{1}{p}} - 1| \leq |y_n - 1| \text{ for all } n \in \mathbf{N},$$

so

$$\begin{aligned} \{y_n\} \to 1 &\implies |y_n - 1| \to 0 \\ &\implies |(y_n)^{\frac{1}{p}} - 1| \to 0 \text{ (by comparison theorem for null sequences)} \\ &\implies \{(y_n)^{\frac{1}{p}}\} \to 1 \\ &\implies \{f_p(y_n)\} \to f_p(1). \end{aligned}$$

Hence f_p is continuous at 1.

Finally we show that f_p is continuous at arbitrary $a \in (0, \infty)$. Let $a \in [0, \infty)$, and let $\{z_n\}$ be a sequence n $[0, \infty)$. Then

$$\begin{aligned} \{z_n\} \to a &\implies \frac{1}{a}\{z_n\} \to \frac{1}{a}a = 1 \\ &\implies \left\{\frac{z_n}{a}\right\} \to 1 \end{aligned}$$

$$\begin{aligned}
&\implies \left\{\left(\frac{z_n}{a}\right)^{\frac{1}{p}}\right\} \to 1 \text{ (since } f_p \text{ is continuous at 1)} \\
&\implies a^{\frac{1}{p}}\left\{\left(\frac{z_n}{a}\right)^{\frac{1}{p}}\right\} \to a^{\frac{1}{p}} \cdot 1 \\
&\implies \{(z_n)^{\frac{1}{p}}\} \to a^{\frac{1}{p}} \\
&\implies \{f_p(z_n)\} \to f_p(a).
\end{aligned}$$

Thus f_p is continuous at a. ‖

8.15 Definition (Composition of functions.) Let A, B, C, D be sets, and let $f: A \to B$, $g: C \to D$ be functions. We define a function $g \circ f$ by the rules:

$$\begin{aligned}
\text{domain } (g \circ f) &= \{x \in \text{dom} f : f(x) \in \text{dom}(g)\} \\
(g \circ f)(x) &= g\,(f(x)) \text{ for all } x \in \text{dom}(g \circ f).
\end{aligned}$$

8.16 Examples. Let $f: \mathbf{C} \to \mathbf{C}$, $g: \mathbf{C} \to \mathbf{C}$ be defined by

$$\begin{aligned}
f(z) &= z^2 + 1 \text{ for all } z \in \mathbf{C} \\
g(z) &= (1 + z^*) \text{ for all } z \in \mathbf{C}.
\end{aligned}$$

Then

$$\begin{aligned}
(f \circ g)(z) &= f\,(g(z)) = (1 + z^*)^2 + 1 = 1 + 2z^* + (z^*)^2 + 1 \\
&= 2 + 2z^* + (z^*)^2,
\end{aligned}$$

and

$$(g \circ f)(z) = g\,(f(z)) = 1 + (z^2 + 1)^* = 1 + (z^*)^2 + 1 = 2 + (z^*)^2.$$

If $f: \mathbf{R} \to \mathbf{R}$ and $g: [-1, \infty) \to \mathbf{R}$ are defined by

$$f(x) = x^2 - 1 \text{ for all } x \in \mathbf{R}$$

and

$$g(x) = \sqrt{1 + x} \text{ for all } x \in [-1, \infty),$$

then

$$\begin{aligned}
(f \circ g)(x) &= (\sqrt{1 + x})^2 - 1 \text{ for all } x \in [-1, \infty) \\
&= 1 + x - 1 \text{ for all } x \in [-1, \infty) \\
&= x \text{ for all } x \in [-1, \infty)
\end{aligned}$$

and

$$(g \circ f)(x) = \sqrt{1 + (x^2 - 1)} = \sqrt{x^2} = |x| \text{ for all } x \in \mathbf{R}.$$

8.17 Theorem (Compositions of continuous functions.) *Let f, g be complex functions. If f is continuous at $a \in \mathbf{C}$, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.*

Proof: Let $\{x_n\}$ be a sequence in $\mathrm{dom}(g \circ f)$ such that $\{x_n\} \to a$. Then for all $n \in \mathbf{N}$, we have $x_n \in \mathrm{dom}(f)$ and $f(x_n) \in \mathrm{dom}(g)$. By continuity of f at a, $\{f(x_n)\} \to f(a)$, and by continuity of g at $f(a)$, $\{g(f(x_n))\} \to g(f(a))$. $\|$

8.18 Example. If $f(x) = \sqrt{x^2+3}$ for all $x \in \mathbf{R}$, then f is continuous (i.e., f is continuous at a for all $a \in \mathbf{R}$.)

8.19 Exercise. Let $f\colon \mathbf{N} \to \mathbf{C}$ be defined by $f(n) = n!$ for all $n \in \mathbf{N}$. Is f continuous?

8.3 Limits

8.20 Definition (Limit point.) Let S be a subset of $\mathbf{C}$ and let $a \in \mathbf{C}$. We say a is a *limit point* of S if there is a sequence f in $S \backslash \{a\}$ such that $f \to a$.

8.21 Example. Let $D(0,1) = \{z \in \mathbf{C}\colon |z| < 1\}$ be the unit disc, and let $\alpha \in \mathbf{C}$. We'll show that α is a limit point of $D(0,1)$ if and only if $|\alpha| \leq 1$.

Proof that (α is a limit point of $D(0,1)$) $\Longrightarrow |\alpha| \leq 1$.
Suppose α is a limit point of $D(0,1)$. Then there is a sequence $\{a_n\}$ in $D(0,1) \setminus \{\alpha\}$ such that $\{a_n\} \to \alpha$. Since the absolute value function is continuous, it follows that $\{|a_n|\} \to |\alpha|$. Since $a_n \in D(0,1)$ we know that $|a_n| < 1$ (and hence $|a_n| \leq 1$.) for all $n \in \mathbf{N}$. By the inequality theorem for limits of sequences, $\lim\{|a_n|\} \leq 1$, i.e. $|\alpha| \leq 1$.

Proof that $(|\alpha| \leq 1) \Longrightarrow \alpha$ is a limit point of $D(0,1)$.

Case 1: Suppose $0 < |\alpha| \leq 1$. Let $f_\alpha(n) = \dfrac{n}{n+1}\alpha$ for all $n \in \mathbf{Z}_{\geq 1}$. Then $|f_\alpha(n)| = \dfrac{n}{n+1}|\alpha| \leq \dfrac{n}{n+1} < 1$ so $f_\alpha(n) \in D(0,1)$, and clearly $f_\alpha(n) \neq \alpha$. Now $\{f_\alpha(n)\}_{n\geq 1} = \left\{\dfrac{1}{1+\frac{1}{n}} \cdot \alpha\right\}_{n \geq 1} \to \alpha$, so α is a limit point of $D(0,1)$.

Case 2: $\alpha = 0$. This case is left to you.

8.22 Exercise. Supply the proof for Case 2 of example 8.21; i.e., show that 0 is a limit point of $D(0,1)$.

8.23 Example. The set $\mathbf{Z}$ has no limit points. Suppose $\alpha \in \mathbf{C}$, and there is a sequence f in $\mathbf{Z}\backslash\{\alpha\}$ such that $f \to \alpha$. Let $g(n) = f(n) - f(n+1)$ for all $n \in \mathbf{N}$. By the translation thoerem $g \to \alpha - \alpha = 0$; i.e., g is a null sequence. Let N_g be a precision function for g. Then for all $n \in \mathbf{N}$,

$$\begin{aligned} n \geq N_g\left(\frac{1}{2}\right) &\implies |g(n)| < \frac{1}{2} \\ &\implies |f(n) - f(n+1)| < \frac{1}{2}. \end{aligned}$$

Now $|f(n) - f(n+1)| \in \mathbf{N}$, so it follows that

$$n \geq N_g\left(\frac{1}{2}\right) \implies |f(n) - f(n+1)| = 0 \implies f(n) = f(n+1)$$

and hence

$$\alpha = \lim f = f\left(N_g\left(\frac{1}{2}\right)\right).$$

This contradicts the fact that $f(n) \in \mathbf{Z}\backslash\{\alpha\}$ for all $n \in \mathbf{N}$. $\|$

8.24 Definition (Limit of a function.) Let f be a complex function, and let a be a limit point of $\text{dom}(f)$. We say that f *has a limit at* a or that $\lim_a f$ *exists* if there exists a function F with $\text{dom}(F) = \text{dom}(f) \cup \{a\}$ such that $F(z) = f(z)$ for all $z \in \text{dom}(f)\backslash\{a\}$, and F is continuous at a. In this case we denote the value of $F(a)$ by $\lim_a f$ or $\lim_{z \to a} f(z)$. Theorem 8.30 shows that this definition makes sense. We will give some examples before proving that theorem.

8.25 Warning. Notice that $\lim_a f$ is defined only when a is a limit point of $\text{dom}(f)$. For each complex number β, define a function $F_\beta : \mathbf{N} \cup \{\frac{1}{2}\} \to \mathbf{C}$ by

$$F_\beta(n) = \begin{cases} n! & \text{if } n \in \mathbf{N}, \\ \beta & \text{if } n = \frac{1}{2}, \end{cases}$$

Then F_β is continuous, and $F(n) = n!$ for all $n \in \mathbf{N}$. If I did not put the requirement that a be a limit point of $\text{dom}(f)$ in the above definition, I'd have

$$\lim_{n \to \frac{1}{2}} n! = F_\beta(\frac{1}{2}) = \beta \text{ for all } \beta \in \mathbf{C}.$$

I certainly do not want this to be the case.

8.26 Example. Let $f(z) = \dfrac{z^2-1}{z-1}$ for all $z \in \mathbf{C}\backslash\{1\}$ and let $F(z) = z+1$ for all $z \in \mathbf{C}$. Then $f(z) = F(z)$ on $\mathbf{C}\backslash\{1\}$ and F is continuous at 1. Hence $\lim\limits_{1} f = F(1) = 2$.

8.27 Example. If $f(z) = \begin{cases} z & \text{for} \neq 1 \\ 3 & \text{for } z = 1 \end{cases}$, then $\lim\limits_{1} f = 1$, since the function $F : z \mapsto z$ agrees with f on $\mathbf{C}\backslash\{1\}$ and is continuous at 1.

8.28 Example. If f is continuous at a, and a is a limit point of domain f, then f has a limit at a, and

$$\lim_{a} f = f(a).$$

8.29 Example. Let $f(z) = \dfrac{z^*}{z}$ for all $z \in \mathbf{C}\setminus\{0\}$. Then f has no limit at 0.

Proof: Suppose there were a continuous function F on $\mathbf{C}$ such that $F(z) = f(z)$ on $\mathbf{C}\backslash\{0\}$. Let $\{a_n\} = \left\{\dfrac{i}{n+1}\right\}$ and $\{b_n\} = \left\{\dfrac{1}{n+1}\right\}$. Then $\{a_n\} \to 0$ and $\{b_n\} \to 0$ and so

$$F(0) = \lim\{F(a_n)\} = \lim\left\{\frac{\frac{-i}{n+1}}{\frac{i}{n+1}}\right\} = \lim\{-1\} = -1$$

and also

$$F(0) = \lim\{F(b_n)\} = \lim\left\{\frac{\frac{1}{n+1}}{\frac{1}{n+1}}\right\} = \lim\{1\} = 1.$$

Hence we get the contradiction $-1 = 1$. |||

8.30 Theorem (Uniqueness of limits.) *Let f be a complex function, and let a be a limit point of* $\mathrm{dom}(f)$. *Suppose F, G are two functions each having domain* $\mathrm{dom}(f) \cup \{a\}$, *and each continuous at a, and satisfying $f(z) = F(z) = G(z)$ for all $z \in$* $\mathrm{dom}(f)\backslash\{a\}$. *Then $F(a) = G(a)$.*

Proof: $F - G$ is continuous at a, and $F - G = 0$ on $\mathrm{dom}(f)\backslash\{a\}$. Let $\{a_n\}$ be a sequence in $\mathrm{dom}(f)\backslash\{a\}$ such that $\{a_n\} \to a$. Since $F - G$ is continuous at a, we have

$$\{(F-G)(a_n)\} \to (F-G)(a);$$

i.e.,

$$\{0\} \to F(a) - G(a),$$

so $F(a) - G(a) = 0$; i.e., $F(a) = G(a)$. $\|$

8.31 Exercise. Investigate the following limits. (Give detailed reasons for your answers). In this exercise you should not conclude from the fact that I've written $\lim_{w \to b} f(w)$ that the implied limit exists.

a) $\lim_{t \to 4} t^{\frac{1}{2}}$.

b) $\lim_{n \to 2} n!$.

c) $\lim_{z \to 0} |z|^2 \left(\frac{1}{z} - \frac{1}{z^*}\right)$.

d) $\lim_{z \to a} \frac{\frac{1}{z} - \frac{1}{a}}{z - a}$. (Here $a \in \mathbf{C} \setminus \{0\}$).

e) $\lim_{t \to 0} \frac{\sqrt{t+4} - 2}{t}$.

8.32 Theorem. *Let f be a complex function and let a be a limit point of* $\text{dom}(f)$. *Then f has a limit at a if and only if there exists a number L in* $\mathbf{C}$ *such that for every sequence y in* $\text{dom}(f)\backslash\{a\}$

$$y \to a \implies f \circ y \to L. \tag{8.33}$$

In this case, $L = \lim_a f$.

Proof: Suppose f has a limit at a, and let F be a continuous function with $\text{dom}(F) = \text{dom}(f) \cup \{a\}$, and $F(z) = f(z)$ for all $z \in \text{dom}(f)\backslash\{a\}$. Let y be a sequence in $\text{dom}(f)\backslash\{a\}$ such that $\{y_n\} \to a$. Then y is a sequence in $\text{dom}(F)$, so by continuity of F,

$$\{f(y_n)\} = \{F(y_n)\} \to F(a).$$

Hence, condition (8.33) holds with $L = F(a)$.

Conversely, suppose there is a number L such that

$$\text{for every sequence } y \text{ in } \text{dom}(f)\backslash\{a\},\ (y \to a \implies f \circ y \to L.). \tag{8.34}$$

Define $F\colon \mathrm{dom}(f)\cup\{a\}\to\mathbf{C}$ by

$$F(z)=\begin{cases} f(z) & \text{if } z\in\mathrm{dom}(f)\backslash\{a\}\\ L & \text{if } z=a.\end{cases}$$

I need to show that F is continuous at a. Let z be a sequence in $\mathrm{dom}(F)$ such that $z\to a$. I want to show that $F\circ z\to L$.

Let w be a sequence in $\mathrm{dom}(f)\setminus\{a\}$ such that $w\to a$. (Such a sequence exists because a is a limit point of $\mathrm{dom}(f)$). Define a sequence y in $\mathrm{dom}(f)\backslash\{a\}$ by

$$y(n)=\begin{cases} z(n) & \text{if } z(n)\neq a\\ w(n) & \text{if } z(n)=a.\end{cases}$$

Let $N_{z-\tilde{a}}$ and $N_{w-\tilde{a}}$ be precision functions for $z-\tilde{a}$ and $w-\tilde{a}$ respectively. Let

$$M(\varepsilon)=\max(N_{z-\tilde{a}}(\varepsilon),N_{w-\tilde{a}}(\varepsilon)) \text{ for all } \varepsilon\in\mathbf{R}^+.$$

Then M is a precision function for $y-\tilde{a}$, since for all $\varepsilon\in\mathbf{R}^+$ and all $n\in\mathbf{N}$,

$$\begin{aligned}&n\geq M(\varepsilon)\\ &\implies \begin{cases} n\geq N_{z-\tilde{a}}(\varepsilon)\implies |z(n)-a|<\varepsilon\implies |y(n)-a|<\varepsilon & \text{if } z(n)\neq a\\ n\geq N_{w-\tilde{a}}(\varepsilon)\implies |w(n)-a|<\varepsilon\implies |y(n)-a|<\varepsilon & \text{if } z(n)=a.\end{cases}\end{aligned}$$

Hence $y\to a$, and by assumption (8.34), it follows that $f\circ y\to L$. I now claim that $F\circ z\to L$, and in fact any precision function P for $f\circ y-\tilde{L}$ is a precision function for $F\circ z-\tilde{L}$. For all $\varepsilon\in\mathbf{R}^+$ and all $n\in\mathbf{N}$,

$$\begin{aligned}&n\geq P(\varepsilon)\implies |f(y(n))-L|<\varepsilon\\ &\quad\implies \begin{cases} |F(z(n))-L|=|f(y(n))-L|<\varepsilon & \text{if } z(n)\neq a\\ |F(z(n))-L|=|F(a)-L|=0<\varepsilon & \text{if } z(n)=a.\end{cases}\end{aligned}$$

This completes the proof. ‖

8.35 Example. Let

$$f(z)=f(x+iy)=f((x,y))=\frac{xy|x|}{x^4+y^2}+iy \text{ for all } z\in\mathbf{C}\backslash\{0\}.$$

I want to determine whether f has a limit at 0, i.e., I want to know whether there is a number L such that for every sequence z in $\mathbf{C}\backslash\{0\}$

$$z\to 0\implies f(z)\to L.$$

If $x \in \mathbf{R}^+$ and $\gamma \in \mathbf{Q}^+$ then

$$f((x, x^\gamma)) = \frac{x \cdot x \cdot x^\gamma}{x^4 + x^{2\gamma}} + ix^\gamma = \begin{cases} \dfrac{x^{\gamma+2}}{x^4(1 + x^{2(\gamma-2)})} + ix^\gamma = \dfrac{x^{\gamma-2}}{1 + x^{2(\gamma-2)}} + ix^\gamma \\ \dfrac{x^{\gamma+2}}{x^{2\gamma}(x^{2(2-\gamma)} + 1)} + ix^\gamma = \dfrac{x^{2-\gamma}}{x^{2(2-\gamma)} + 1} + ix^\gamma \end{cases}$$

Since $|2 - \gamma|$ is either $2 - \gamma$ or $\gamma - 2$, we have

$$f((x, x^\gamma)) = \frac{x^{|2-\gamma|}}{1 + x^{2|2-\gamma|}} + ix^\gamma.$$

For each $\gamma \in \mathbf{Q}^+$, define a sequence z_γ by

$$z_\gamma : n \mapsto \left(\frac{1}{n}, \frac{1}{n^\gamma}\right) \text{ for all } n \in \mathbf{Z}^+.$$

Then $z_\gamma \to 0$, and

$$f(z_\gamma(n)) = \frac{\frac{1}{n^{|2-\gamma|}}}{1 + \frac{1}{n^{2|2-\gamma|}}} + \frac{i}{n^\gamma}.$$

Hence

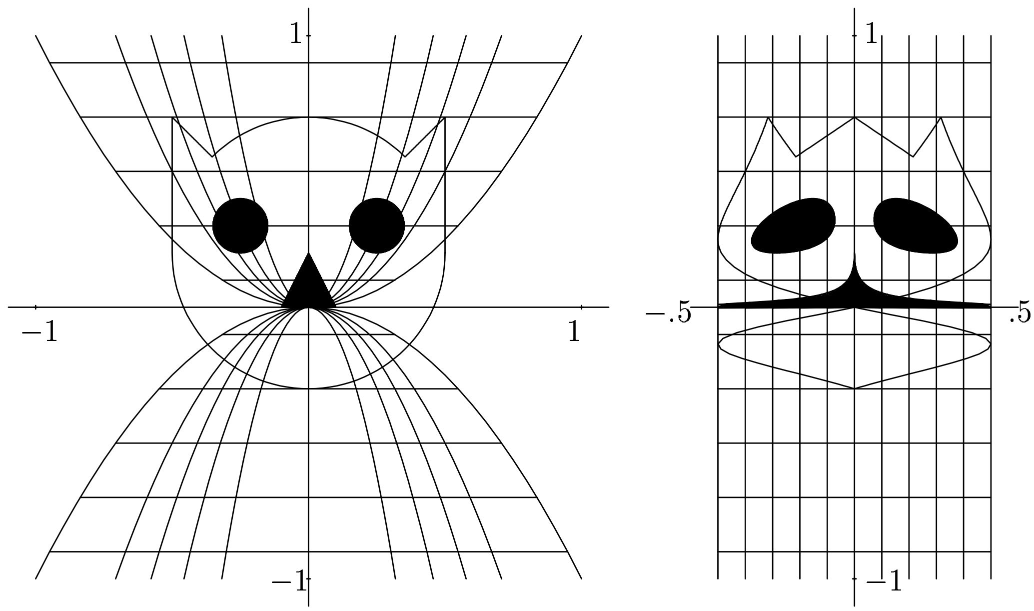
$$\begin{aligned} f \circ z_\gamma &\to 0 \quad \text{if} \quad \gamma \neq 2 \\ f \circ z_\gamma &\to \frac{1}{2} \quad \text{if} \quad \gamma = 2. \end{aligned}$$

It follows that f has no limit at 0.

Let $y_0 \in \mathbf{R}$. It is clear that f maps points on the horizontal line $y = y_0$ to other points on the line $y = y_0$. I'll now look at the image of the parabola $y = cx^2$ under f.

$$f(x + icx^2) = \frac{xcx^2|x|}{x^4 + c^2x^4} + icx^2 = \frac{|x|}{x}\left(\frac{c}{1 + c^2}\right) + icx^2 \text{ for } x \neq 0.$$

So f maps the right half of the parabola $y = cx^2$ into the vertical line $x = \dfrac{c}{1 + c^2}$, and f maps the left half of the parabola to the line $x = \dfrac{-c}{1 + c^2}$. Parabolas with $c > 0$ get mapped to the upper half plane, and parabolas with $c < 0$ get mapped to the lower half plane. The figure below shows some parabolas and horizontal lines and their images under f.



Discontinuous Image of a Cat

8.36 Entertainment. Explain how the cat's nose in the above picture gets stretched, while its cheeks get pinched to a point. (Hint: The figure shows the images of some parabolas $y = cx^2$ where $|c| \geq 1$. What do the images of the parabolas $y = cx^2$ look like when $|c| < 1$?)

8.37 Example. It isn't quite true that "the limit of the sum is the sum of the limits." Let

$$\begin{aligned} f(x) &= \sqrt{x} \text{ for } x \in [0, \infty) \\ g(x) &= \sqrt{-x} \text{ for } x \in (-\infty, 0]. \end{aligned}$$

Then from the continuity of the square root function and the composition theorem,

$$\lim_0 f = 0 = \lim_0 g.$$

But $\lim\limits_0 (f + g)$ does not exist, since $\mathrm{dom}(f + g) = \{0\}$ and 0 is not a limit point of $\mathrm{dom}(f + g)$.

8.38 Theorem (Sum and product theorem.) *Let f, g be complex functions and let a be a limit point of* $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. *If $\lim\limits_a f$ and $\lim\limits_a g$ exist,* *then* $\lim\limits_a(f+g)$, $\lim\limits_a(f-g)$ *and* $\lim\limits_a(f\cdot g)$ *all exist and*

$$\begin{aligned}\lim_a(f+g) &= \lim_a f + \lim_a g,\\ \lim_a(f-g) &= \lim_a f - \lim_a g,\\ \lim_a(f\cdot g) &= \lim_a f\cdot \lim_a g.\end{aligned}$$

If a *is a limit point of* $\mathrm{dom}\left(\dfrac{f}{g}\right)$ *and* $\lim\limits_a g \neq 0$ *then* $\lim\limits_a \dfrac{f}{g}$ *exists and* $\lim\limits_a\left(\dfrac{f}{g}\right) = \dfrac{\lim\limits_a f}{\lim\limits_a g}$.

Proof: Suppose that $\lim\limits_a f$ and $\lim\limits_a g$ exist. Let x be any sequence in $\mathrm{dom}(f+g)\backslash\{a\}$ such that $x \to a$. Then x is a sequence in both $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$, so

$$\lim\{f(x_n)\} = \lim_a f \text{ and } \lim\{g(x_n)\} = \lim_a g.$$

By the sum theorem for limits of sequences,

$$\lim\{(f+g)(x_n)\} = \lim\{f(x_n)\} + \lim\{g(x_n)\} = \lim_a f + \lim_a g.$$

Hence $f+g$ has a limit at a, and $\lim\limits_a(f+g) = \lim\limits_a f + \lim\limits_a g$.

The other parts of the theorem are proved similarly, and the proofs are left to you. $\|$

8.39 Exercise. Prove the product theorem for limits; i.e., show that if f, g are complex functions such that f and g have limits at $a \in \mathbf{C}$, and if a is a limit point of $\mathrm{dom}(f)\cap\mathrm{dom}(g)$, then $f\cdot g$ has a limit at a and

$$\lim_a(f\cdot g) = \lim_a f\cdot\lim_a g.$$

8.40 Definition (Bounded set and function.) A subset S of $\mathbf{C}$ is *bounded* if S is contained in some disc $\bar{D}(0,B)$; i.e., if there is a number B in $\mathbf{R}^+$ such that $|s| \leq B$ for all $s \in S$. We call such a number B a *bound* for S.

Now suppose $f\colon U \to \mathbf{C}$ is a functiom from some set U to $\mathbf{C}$ and A is a subset of U. We say *f is bounded on A* if $f(A)$ is a bounded set, and any bound for $f(A)$ is called a *bound for* f on A. Thus a number $B \in \mathbf{R}^+$ is a bound for f on A if and only if

$$|f(a)| \leq B \text{ for all } a \in A.$$

We say f *is bounded* if f is bounded on dom(f). If f is not bounded on A, we say f *is unbounded on* A.

8.41 Examples. The definition of bounded sequence given in 7.41 is a special case of the definition just given for bounded function.

Let $f(z) = \dfrac{1}{1+z^2}$ for all $z \in \mathbf{C}\backslash\{\pm i\}$. Then f is bounded on $\mathbf{R}$ since

$$|f(z)| \leq \frac{1}{1+z^2} \leq 1 \text{ for all } z \in \mathbf{R}.$$

However, f is not a bounded function, since

$$\begin{aligned} \left|f\left(i+\frac{1}{n}\right)\right| &= \left|\frac{1}{1+\left(-1+\frac{2i}{n}+\frac{1}{n^2}\right)}\right| = \left|\frac{n}{2i+\frac{1}{n}}\right| \\ &= \frac{n}{\sqrt{4+\frac{1}{n^2}}} \geq \frac{n}{\sqrt{5}} \end{aligned}$$

for all $n \in \mathbf{Z}_{\geq 1}$.

Let

$$F(z) = \begin{cases} \dfrac{xy|x|}{x^4+y^2} & \text{for } z \in \mathbf{C}\backslash\{0\} \\ 0 & \text{for } z = 0. \end{cases}$$

(F is the real part of the discontinuous function from example 8.35.)

I claim F is bounded by 1. For all $a, b \in \mathbf{R}$,

$$|a|\ |b| \leq \max(|a|,|b|)^2 \leq a^2 + b^2.$$

(NOTE: $\max(|a|,|b|)^2$ is either a^2 or b^2.) Hence if $(a,b) \neq (0,0)$, then

$$\left|\frac{ab}{a^2+b^2}\right| \leq 1.$$

To prove my claim, apply this result with $a = x|x|$ and $b = y$.

8.42 Exercise. Show that

$$\left|\frac{ab}{a^2+b^2}\right| \leq \frac{1}{2}$$

for all $(a,b) \in \mathbf{R}\times\mathbf{R}\backslash\{(0,0)\}$, and that equality holds if and only if $|a| = |b|$. (This shows that $\frac{1}{2}$ is a bound for the function F in the previous example.) HINT: Consider $(|a| - |b|)^2$.

8.43 Exercise. For each of the functions f below:

1) Decide whether f is bounded, and if it is, find a bound for f.

2) Decide whether f is bounded on $\text{dom}(f)\cap D(0,1)$, and if it is, find a bound for f on $\text{dom}(f)\cap D(0,1)$.

3) Decide whether f has a limit at 0, and if it does, find $\lim\limits_{0} f$.

Here $z = (x,y) = x + iy$.

a) $f(z) = \dfrac{x^2}{x^2+y^2}$ for all $z \in \mathbf{C}\backslash\{0\}$.

b) $f(z) = \dfrac{x^2y}{x^2+y^2}$ for all $z \in \mathbf{C}\backslash\{0\}$.

c) $f(z) = \dfrac{(z^*)^2}{z^2}$ for all $z \in \mathbf{C}\backslash\{0\}$.

d) $f(x) = \dfrac{x^2+x^6}{x^2+x^4}$ for all $x \in \mathbf{R}\backslash\{0\}$.

e) $f(x) = \dfrac{\sqrt{x+1}-1}{x}$ for all $x \in [-1,\infty)\backslash\{0\}$.