Chapter 4

The Complexification of a Field.

Throughout this chapter, F will represent a field in which -1 is not a square. For example, in an ordered field -1 is not a square, but in $\mathbf{Z}_5$, $(2)^2 = 4 = -1$ so -1 is a square. In $\mathbf{Z}_3$,

$$0^2 = 0, \quad 1^2 = 1, \quad 2^2 = 1, \quad \text{and } -1 = 2,$$

so -1 is not a square in $\mathbf{Z}_3$.

Let F be a field in which -1 is not a square. I am going to construct a new field $\mathbf{C}_F$ which contains (a copy of) F and a new element i such that $i^2 = -1$. The elements of $\mathbf{C}_F$ will all have the form

$$a + bi$$

where a and b are in F. I'll call $\mathbf{C}_F$ the *complexification* of F. Before I start my construction, note that if a, b, c, d are in F and $i^2 = -1$, then by the usual field axioms

$$(a + bi) + (c + di) = (a + c) + (b + d)i, \tag{4.1}$$

and

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i. \tag{4.2}$$

4.1 Construction of $\mathbf{C}_F$.

Let F be a field in which -1 is not a square. Let $\mathbf{C}_F = F \times F$ denote the Cartesian product of F with itself (Cf. definition 1.55). I define two binary operations $\oplus$ and $\odot$ on $\mathbf{C}_F$ as follows (cf. (4.1) and (4.2)): for all $(a,b),(c,d) \in \mathbf{C}_F$,

$$(a,b) \oplus (c,d) = (a+c, b+d)$$

and

$$(a,b) \odot (c,d) = (ac - bd, ad + bc).$$

We will now show that $(\mathbf{C}_F, \oplus, \odot)$ is a field.

4.3 Theorem (Associativity of $\odot$.) *The operation $\odot$ is associative on $\mathbf{C}_F$.*

Proof: Let $(a,b),(c,d)$ and (e,f) be elements in $\mathbf{C}_F$. Then

$$\begin{aligned}
&(a,b) \odot ((c,d) \odot (e,f)) \\
&\quad= (a,b) \cdot (ce - df, cf + de) \\
&\quad= (a(ce-df) - b(cf+de), a(cf+de) + b(ce-df)) \\
&\quad= (ace - adf - bcf - bde, acf + ade + bce - bdf). \qquad (4.4)
\end{aligned}$$

Also,

$$\begin{aligned}
&((a,b) \odot (c,d)) \odot (e,f) \\
&\quad= (ac - bd, ad + bc) \odot (e,f) \\
&\quad= ((ac-bd)e - (ad+bc)f, (ac-bd)f + (ad+bc)e) \\
&\quad= (ace - bde - adf - bcf, acf - bdf + ade + bce). \qquad (4.5)
\end{aligned}$$

Now by using the field properties of F, we see that the (4.4) and (4.5) are equal, and hence

$$(a,b) \odot ((c,d) \odot (e,f)) = ((a,b) \odot (c,d)) \odot (e,f).$$

Hence, $\odot$ is associative on $\mathbf{C}_F$. $\|$

I expect the multiplicative identity for $\mathbf{C}_F$ to be $1 + 0i = (1,0)$.

4.6 Theorem (Multiplicative identity for $\mathbf{C}_F$.) *The element $(1,0)$ is an identity for $\odot$ on $\mathbf{C}_F$.*

Proof: For all $(a,b) \in \mathbf{C}_F$, we have

$$(1,0) \odot (a,b) = (1 \cdot a - 0 \cdot b, 1 \cdot b + 0 \cdot a) = (a,b)$$

and

$$(a,b) \odot (1,0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a,b). \; \|$$

4.7 Exercise.

a) Show that $\oplus$ is associative on $\mathbf{C}_F$.

b) Show that there is an identity for $\oplus$ on $\mathbf{C}_F$.

c) Show that every element in $\mathbf{C}_F$ has an inverse for $\oplus$.

d) Show that $\odot$ is commutative on $\mathbf{C}_F$.

e) Show that the distributive law holds for $\mathbf{C}_F$.

f) Show that the additive and multiplicative identities for $\mathbf{C}_F$ are different.

As a result of exercise 4.7 and the two previous theorems, we have verified that $(\mathbf{C}_F, \oplus, \odot)$ satisfies all of the field axioms except existence of multiplicative inverses. Note that up to this point we have never used the assumption that -1 is not a square in F.

4.8 Theorem (Existence of multiplicative inverses.) *Let F be a field in which -1 is not a square and let (a,b) be an element in $\mathbf{C}_F \backslash \{(0,0)\}$. Then (a,b) has an inverse for $\odot$.*

Proof: Let $(a,b) \in \mathbf{C}_F \backslash \{(0,0\}$. I want to find a point $(x,y) \in \mathbf{C}_F$ such that

$$(a,b) \odot (x,y) = (1,0).$$

Since multiplication is commutative, this shows that $(x,y) \odot (a,b) = (1,0)$ and hence that (x,y) is a multiplicative inverse for (a,b). I want

$$(ax - by, ay + bx) = (1,0),$$

so I want

$$bx + ay = 0 \tag{4.9}$$

and

$$ax - by = 1. \tag{4.10}$$

Multiply the first equation by b and the second by a to get

$$\begin{aligned} b^2x + aby &= 0 \\ a^2x - aby &= a. \end{aligned}$$

If we add these equations, we get

$$(a^2 + b^2)x = a. \tag{4.11}$$

In the next lemma I'll show that if -1 is not a square then $a^2 + b^2 \neq 0$ for all $(a, b) \in \mathbf{C}_F \backslash \{(0, 0)\}$, so by (4.11), $x = \dfrac{a}{a^2 + b^2}$. Now multiply (4.9) by a and (4.10) by $-b$ to get

$$\begin{aligned} abx + a^2 y &= 0 \\ -abx + b^2 y &= -b. \end{aligned}$$

If we add these equations, we get

$$(a^2 + b^2)y = -b$$

so

$$y = \frac{-b}{a^2 + b^2}.$$

I've shown that if $(a, b) \odot (x, y) = (1, 0)$, then $(x, y) = \left(\dfrac{a}{a^2 + b^2}, \dfrac{-b}{a^2 + b^2}\right)$. A direct calculation shows that this works:

$$(a, b) \odot \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right) = \left(\frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}, \frac{a(-b) + ba}{a^2 + b^2}\right) = (1, 0). \ \|$$

4.12 Remark. The above proof shows that for all $(a, b) \in \mathbf{C} \setminus \{(0, 0)\}$,

$$(a, b)^{-1} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right).$$

4.13 Lemma. *Let F be a field in which -1 is not a square. Let (a, b) be an element in $\mathbf{C}_F \backslash \{(0, 0)\}$. Then $a^2 + b^2 \neq 0$.*

Proof: Since $(a, b) \neq (0, 0)$, either $a \neq 0$ or $b \neq 0$.

Case 1: Suppose $a \neq 0$, then $a^2 \neq 0$, so

$$\begin{aligned} a^2 + b^2 = 0 \implies & a^2 \left(1 + \left(\frac{b}{a}\right)^2\right) = 0 \\ \implies & 1 + \left(\frac{b}{a}\right)^2 = 0 \\ \implies & \left(\frac{b}{a}\right)^2 = -1. \end{aligned}$$

Since -1 is not a square in F, $a^2 + b^2 \neq 0$.

Case 2: Suppose $b \neq 0$. Repeat the argument of Case 1 with the roles of a and b interchanged. |||

We now have verified all of the field axioms so we know that $\mathbf{C}_F$ is a field. Hence we can calculate in $\mathbf{C}_F$ using all of the algebraic results that have been proved to hold in all fields.

4.14 Notation $(i, \tilde{a})$ Let F be a field in which -1 is not a square. We will denote the pair $(0,1) \in \mathbf{C}_F$ by i, and if $a \in F$ we will denote the pair $(a,0) \in \mathbf{C}_F$ by $\tilde{a}$.

We have $\tilde{0} = (0,0)$ is the additive identity for $\mathbf{C}_F$, and $\tilde{1} = (1,0)$ is the multiplicative identity for $\mathbf{C}_F$. If $a \in F$, then $-\tilde{a} = -(a,0) = (-a,0) = \widetilde{-a}$. Also

$$i^2 = (0,1) \odot (0,1) = (0-1,0) = (-1,0) = -\tilde{1},$$

so i is a square root of $-\tilde{1}$.

If $a, b \in F$, then

$$\begin{aligned} \tilde{a} \oplus (\tilde{b} \odot i) &= (a,0) \oplus ((b,0) \odot (0,1)) \\ &= (a,0) \oplus (0,b) = (a,b), \end{aligned}$$

and hence every element $(a,b) \in \mathbf{C}_F$ can be written in the form $\tilde{a} \oplus (\tilde{b} \odot i)$. We have

$$\begin{aligned} \tilde{a} \odot \tilde{b} &= (a,0) \odot (b,0) = (ab,0) = \widetilde{ab} \\ \tilde{a} \oplus \tilde{b} &= (a,0) \oplus (b,0) = (a+b,0) = \widetilde{a+b}. \end{aligned}$$

Hence $\mathbf{C}_F$ contains a "copy of F". Each element a in F corresponds to a unique $\tilde{a}$ in $\mathbf{C}_F$ in such a way that addition in $\mathbf{C}_F$ corresponds to addition in F and multiplication in $\mathbf{C}_F$ corresponds to multiplication in F. We will henceforth drop the tildes, and we'll denote $\oplus$ by $+$ and $\odot$ by $\cdot$ as is usual in fields. Then every element in $\mathbf{C}_F$ can be written uniquely as $a + bi$ where $a, b \in F$ and $i^2 = -1$.

We consider F to be a subset of $\mathbf{C}_F$. An element $z = (a,b) = a + bi$ of $\mathbf{C}_F$ is in F if and only if $b = 0$. If $a, b, c, d \in F$, then

$$a + bi = c + di \iff (a,b) = (c,d) \iff a = c \text{ and } b = d.$$

4.15 Examples. I will find the square roots of $2i$ in $\mathbf{C_Q}$. Let $a, b \in \mathbf{Q}$. Then

$$\begin{aligned}(a+bi)^2 = 2i &\iff a^2 - b^2 + 2abi = 2i \\ &\iff a^2 - b^2 = 0 \text{ and } 2ab = 2 \\ &\iff a^2 = b^2 \text{ and } ab = 1 \\ &\iff (a = b \text{ and } ab = 1) \text{ or } (a = -b \text{ and } ab = 1).\end{aligned}$$

Now

$$(a = b \text{ and } ab = 1) \iff (a = b \text{ and } a^2 = 1) \iff a + bi = \pm(1+i)$$

and

$$(a = -b \text{ and } ab = 1) \implies -b^2 = 1 \implies b^2 = -1$$

which is impossible. The only possible square roots of $2i$ are $\pm(1+i)$. You can easily verify that these are square roots of $2i$.

4.16 Example. I can solve the quadratic equation

$$z^2 - 4z + 4 - \frac{1}{2}i = 0 \tag{4.17}$$

in $\mathbf{C_Q}$ by using the quadratic formula for $Az^2 + Bz + C = 0$.
$B^2 - 4AC = 16 - 4\left(4 - \frac{1}{2}i\right) = 2i = (1+i)^2$ (by the previous example). Since $B^2 - 4AC$ is a square, the equation has the solution set

$$\left\{\frac{4+(1+i)}{2}, \frac{4-(1+i)}{2}\right\} = \left\{\frac{5}{2} + \frac{1}{2}i, \frac{3}{2} - \frac{1}{2}i\right\}.$$

4.18 Exercise. Check that $\frac{5}{2} + \frac{1}{2}i$ and $\frac{3}{2} - \frac{1}{2}i$ are solutions to (4.17).

4.19 Exercise.

a) Write $\dfrac{1}{1-2i}$ in the form $a + bi$ where $a, b \in \mathbf{Q}$.

b) Find all solutions to

$$(1-2i)z^2 - 2z + 1 = 0$$

in $\mathbf{C_Q}$. (You may want to use the result of example 4.15.) Write your solutions in the form $a + bi$ where $a, b \in \mathbf{Q}$.

4.20 Entertainment. We noted earlier that -1 is not a square in $\mathbf{Z}_3$, so $\mathbf{Z}_3$ has a complexification, which is a field with 9 elements. Show that if $z = 1 + i$, then the 9 elements in $\mathbf{C}_{\mathbf{Z}_3}$ are

$$\{0, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8\}.$$

Can you figure out before you make any calculations which of these elements is 1?

4.2 Complex Conjugate.

4.21 Definition (Complex conjugate.) Let F be a field in which -1 is not a square. Let $z = (a, b) = a + bi$ be an element of $\mathbf{C}_F$. We define

$$z^* = (a, -b) = a - bi.$$

z^* is called the *conjugate* of z.

The following remark will be needed somewhere in the proof of the next exercise.

4.22 Remark. If F is a field in which -1 is not a square, then $2 \neq 0$ in F, since

$$\begin{aligned} 2 = 0 &\implies 1 + 1 = 0 \\ &\implies -1 = 1 \\ &\implies -1 = 1^2 \\ &\implies -1 \text{ is a square .} \end{aligned}$$

4.23 Exercise. Let F be a field in which -1 is not a square. Let $z, w \in \mathbf{C}_F$. Show that

a) $(z + w)^* = z^* + w^*$.

b) $(z \cdot w)^* = z^* \cdot w^*$.

c) $\left(\dfrac{z}{w}\right)^* = \dfrac{z^*}{w^*}$ if $w \neq 0$.

d) $z^* = 0 \iff z = 0$.

e) If $z = a + bi \in \mathbf{C}_F$, then $zz^* = a^2 + b^2 \in F$. If $z \neq 0$ then $zz^* \neq 0$.

f) $z^* = z \iff z \in F$.

g) $z^{**} = z$.

4.24 Example. The results of the previous exercise provide a way to write expressions of the form $\frac{z}{w}$ in the form $a + bi$. Write

$$\frac{z}{w} = \frac{z}{w} \cdot \frac{w^*}{w^*}$$

and calculate away. For example, in $\mathbf{C}_{\mathbf{Q}}$, we have

$$\begin{aligned} \frac{2+i}{(3-i)(4+5i)} &= \frac{(2+i)}{(3-i)(4+5i)} \cdot \frac{(3+i)(4-5i)}{(3+i)(4-5i)} \\ &= \frac{(2+i)(17-11i)}{(3^2+1^2)(4^2+5^2)} = \frac{45-5i}{10 \cdot 41} = \frac{5(9-i)}{5 \cdot 82} \\ &= \frac{9}{82} - \frac{1}{82}i. \end{aligned}$$

4.25 Exercise. Write each of the following elements of $\mathbf{C}_{\mathbf{Q}}$ in the form $a+bi$ where $a, b \in \mathbf{Q}$.

a) $\dfrac{(4-2i)(1+2i)}{(1-3i)(-1+3i)}$

b) $(1+i)^{10}$

4.26 Note. The first appearance of complex numbers is in *Ars Magna* (1545) by Girolamo Cardano (1501-1576).

> If it should be said, Divide 10 into two parts the product of which is 30 or 40, it is clear that this case is impossible. Nevertheless, we will work thus: $\cdots$ [16, page 219].

He then proceeds to calculate that the parts are $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$, and says

> Putting aside the mental tortures involved, multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$ making $25 - (-15)$ which is $+15$. Hence this product is $40 \cdots$. So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless [16, page 219–220].

Around 1770, Euler wrote

> 144. All such expressions as $\sqrt{}$-1, $\sqrt{}$-2, $\sqrt{}$-3, $\sqrt{}$-4 &c are consequently impossible, or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing nor greater than nothing, nor less than nothing; which necessarily constitutes them imaginary, or impossible.
>
> 145. But notwithstanding this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by $\sqrt{}$-4 is meant a number which, multiplied by itself, produces -4; for this reason also, nothing prevents us from making use of these imaginary numbers, and employing them in calculation. [20, p 43]

The use of the letter i to represent $\sqrt{-1}$ was introduced by Euler in 1777.[15, vol 2, p 128] Both Maple and Mathematica use `I` to denote $\sqrt{-1}$.

The first attempts to "justify" the complex numbers appear around 1800. The early descriptions were geometrical rather than algebraic. The algebraic construction of $\mathbf{C}_F$ used in these notes follows the ideas described by William Hamilton circa 1835 [25, page 83].

You will often find the complex conjugate of z denoted by $\overline{z}$ instead of z^*. The notion of complex conjugate seems to be due to Cauchy[45, page 26], who called $a + bi$ and $a - bi$ conjugates of each other.