Appendix A

Hints and Answers

Exercise 3: The Rhind value is \( \frac{256}{81} = 3.1604 \ldots \)

Exercise 1.7: Look at the boundary.

Exercise 1.10: If a set has no endpoints, then it contains all of its endpoints and none of its endpoints.

Exercise 2.10: \( 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \).

Exercise 2.18: \( \text{area} T(a) = \frac{2}{3} a^\frac{3}{2} \).

Exercise 2.27: \( .027027027 \ldots = \frac{1}{37} \).

Exercise 2.36: I let \( O_j = B(a^{\frac{j}{N}}, a^{\frac{j-1}{N}}; 0, a^{-\frac{2j}{N}}) \) and \( I_j = B(a^{\frac{j}{N}}, a^{\frac{j-1}{N}}; 0, a^{-\frac{2(j-1)}{N}}) \).

Exercise 3.20: Recall \( (R \implies S) \iff (S \text{ or not } R) \).

Exercise 5.61: \( S_{ab}[\frac{1}{1}] = S_a[\frac{1}{1}] \cup S_{ab}[\frac{1}{1}] \). (Draw a picture.)

Exercise 5.80: Consider a partition with a fairly large number of points.

Exercise 6.33: The assertion is false.

Exercise 6.59: (part e) The limit is \( \frac{1}{3} \). It simplifies matters if you factor both the numerator and the denominator. The sequence in part g) is a translate of the sequence in part f).

Exercise 6.69: All four statements are false.
Exercise 6.94: a) \((1 + \frac{3}{n})^{2n} = ((1 + 3/n)^n)^2\).

Exercise 6.97: \((1 - \frac{e}{n})^n = \frac{(1 - \frac{e^2}{n^2})^n}{(1 + \frac{e^2}{n^2})^n}\).

Exercise 7.16: Take \(c = \frac{b}{a}\) in lemma 7.13.

Exercise 7.18: \(A_{\alpha}^nf = A_{\beta}^1f + A_{\alpha}^1f\). Show that \(A_{\alpha}^1f\) is small when \(n\) is large.

Exercise 8.14: d) \(\frac{x+1}{x} = 1 + \frac{1}{x}\). Not all of these integrals exist.

Exercise 8.16: Show that \(\sum(f, P, S) \leq \sum(g, P, S)\) for every partition \(P\) of \([a, b]\) and every sample \(S\) for \(P\).

Exercise 8.28: \(g\) is the sum of an integrable function and a spike function.

Exercise 8.32: \(f\) is not piecewise monotonic. It is easy to see that \(f\) is integrable on \([1, 2]\). If you can show it is integrable on \([0, 1]\) then you are essentially done.

Exercise 8.34: b) \((b - a)^{3/6}\).

Exercise 8.41: For any partition \(P\) of \([0, 1]\) you can find a sample \(S\) such that \(\sum(R, P, S) = 0\).

Exercise 8.46: In equation 8.44, replace \(r\) by \(\frac{1}{R}\), and replace \(a\) and \(b\) by \(RA\) and \(RB\).

Exercise 8.48: \(\alpha(E_{ab}) = \pi ab\).

Exercise 8.50: If \(a = 1/4\) then both areas are approximately 3.1416

Exercise 8.55: area = \(4\pi\).

Exercise 8.57: The areas are \(5/12\) and 1.

Exercise 8.58: The area is \(\frac{37}{12}\). Some fractions with large numerators may appear along the way.

Exercise 9.20: The last two formulas are obtained from the second by replacing \(t\) by \(t/2\).
Exercise 9.29: I used exercise 9.28 with \( x = \frac{\pi}{6} \) to find \( \cos \left( \frac{\pi}{6} \right) \). You can also give a more geometric proof.

Exercise 9.44: You will need to use (9.24).

Exercise 9.48: \( \alpha(S_0^\pi(\sin)) = 2 \).

Exercise 9.49: \( \text{area} = \sqrt{2} \).

Exercise 9.69: \[
\int_{\pi/2}^{0} \sin(x) \, dx = 1; \int_{\pi/2}^{0} \sin^2(x) \, dx = \pi/4; \int_{\pi/2}^{0} \sin^4(x) \, dx = 3\pi/16.
\]

Exercise 10.25: \( f'(a) = -\frac{1}{a^2} \).


Exercise 10.27: \( f'(a) = \frac{1}{(a+1)^2} \).

Exercise 10.28: \( y = 2x - 4; y = -6x - 4 \).

Exercise 11.6: I used formula 9.25.

Exercise 11.15: \( \frac{d}{dt}(|-100t|) = \frac{100u}{|u|} \).

Exercise 11.21: You can use the definition of derivative, or you can use the product rule and the reciprocal rule.

Exercise 11.24: \( f'(x) = \ln(x), g'(x) = \frac{ad-bc}{(cx+d)^2}, k'(x) = 2(2x+3)(x^2+3x+11) \)

Exercise 11.29: \( (g \circ (g \circ g))(x) = ((g \circ g) \circ g)(x) = x \) for \( x \in \mathbb{R} \setminus 0, 1 \). If you said \( (f \circ f)(x) = x \), calculate both sides when \( x = -1 \).

Exercise 11.40: Use the definition of derivative. \( h'(2) = 0 \).

Exercise 11.43: \( g'(x) = -\tan(x), h'(x) = \tan(x), k'(x) = \sec(x), l'(x) = -\csc(x), m'(x) = 9x^2 \ln(5x), n'(x) = \frac{\sqrt{x^4+1}}{x} \) (It requires a lot of calculation to simplify \( n' \)), \( p'(x) = \frac{x^2}{x^2+1}, q'(x) = \sin(\ln(|6x|)) \).

Exercise 12.14: d) Such a function \( k \) does exist.

Exercise 12.15: a) Use extreme value property.

Exercise 12.27: Proof is like given proof of corollary 12.26.

Exercise 12.31: Apply corollary 12.26 to \( F - G \).
Exercise 12.35: Yes.

Exercise 12.36: You can apply the chain rule to the identity \( f(-x) = f(x) \).

Exercise 13.14: The function to minimize is \( f(x) = \text{distance}((0, \frac{3}{2}), (x, x^2)) \).

Exercise 13.15: You may get a complicated equation of the form \( \sqrt{f(x)} = \sqrt{g(x)} \) to solve. Square both sides and the equation should simplify.

Exercise 14.5: Apply the intermediate value property to \( f - fp \).

Exercise 14.9: One of the zeros is in \([1, 2]\).

Exercise 14.10: I showed that if \( \text{temp}(A) < \text{temp}(B) < \text{temp}(D) \), then there is a point \( Q \) in \( DC \cup CA \) such that \( \text{temp}(Q) = \text{temp}(B) \).

Exercise 14.11: If \( \text{temp}(A) < \text{temp}(B) < \text{temp}(C) < \text{temp}(D) \), find two points different from \( B \) that have the same temperature as \( B \).

Exercise 14.17: You may want to define some of these functions using more than one formula.

Exercise 14.41: Use the extreme value property to get \( A \) and \( B \).

Exercise 14.54: \( f'(x) = 2\sqrt{a^2 - x^2}; h'(x) = \arccos(ax); n'(x) = (a^2 + b^2)e^{ax}\sin(bx); p'(x) = a^3x^2e^{ax} \).

Exercise 14.55: It is not true that \( l(x) = x \) for all \( x \). Note that the image of \( l \) is \([\frac{-\pi}{2}, \frac{\pi}{2}]\).

Exercise 15.5: \( g^{(k)}(t) = tf^{(k)}(t) + kf^{(k-1)}(t) \).

Exercise 15.8: Use the antiderivative theorem twice.

Exercise 15.9: \( (fg)^{(3)} = fg^{(3)} + 3f^{(1)}g^{(2)} + 3f^{(2)}g^{(1)} + f^{(3)}g \).

Exercise 15.13: You will need to use a few trigonometric identities, including the reflection law (9.18).

Exercise 15.22: \( h(t) = h_0 + v_0t - \frac{1}{2}gt^2 \).

Exercise 16.2: You probably will not be able to find a “single formula” for this. My function has a local maximum at $\frac{1}{2n+1}$ for all $n \in \mathbb{Z}^+$.

Exercise 16.8: The result is known if $p < q$. To get the result when $q < p$, apply 16.6 to $f$ on $[q, p]$.

Exercise 16.13: Not all of these integrals make sense. $K'(x) = 1$ for all $x \in \mathbb{R}$. $L'(x) = 1$ for $x \in \mathbb{R}^+$. $L'(x) = -1$ for $x \in \mathbb{R}^-$. $L(0)$ is not defined.

Exercise 17.16: b) $-\ln(|\csc(e^x) + \cot(e^x)|)$; f) $\frac{1}{2} \ln(|\sin(2x)|)$. i) Cf example 9.68. j) You did this in exercise 9.69.

Exercise 17.31: b) $\frac{1}{2} e^x (\sin(x) - \cos(x))$. When you do the second integration by parts, be careful not to undo the first. c) $x \arctan(x) - \frac{1}{2} \ln(1 + x^2)$. Let $g'(x) = 1$. d) and e) can be done easily without using integration by parts. f) If $r = -1$ the answer is $\frac{1}{2} (\ln(|x|))^2$.

Exercise 17.42: c) Let $u = \sqrt{x}$. You will need an integration by parts. d) Let $u = \ln(3x)$. e) Remember the definition of $2^x$.

Exercise 17.49: a) $\frac{1}{4} a^2 \arcsin\left(\frac{x}{a}\right) + \frac{1}{2} x \sqrt{a^2 - x^2}$. If you forget how to find $\int \cos^2(\theta) d\theta$, review example 9.53. Also recall that $\sin(2x) = 2 \sin(x) \cos(x)$. b) $\ln\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right)$. c) and d) do not require a trigonometric substitution.

Exercise 17.53: a) $\frac{5}{4}$. b) $\frac{\pi}{6}$. c) $\frac{4}{15}$.

Exercise 17.54: $\frac{2\pi}{3} + \sqrt{3}$.

Exercise 17.64: (g) $\ln\left(\sqrt{x^2 + 2x + 2} + x + 1\right)$. First complete the square, and then reduce the problem to $\int \frac{1}{\sqrt{u^2 + 1}} du$. 