Chapter 17

Antidifferentiation Techniques

17.1 The Antidifferentiation Problem

17.1 Definition ($\int f$ or $\int f(x)dx$.) I am going to use the notation $\int f$ or $\int f(x)dx$ to denote some arbitrary antiderivative for f on an interval that often will not be specified. This is the same notation that I used previously to denote an indefinite integral for f. Although the fundamental theorem of the calculus shows that for nice functions the concepts of "antiderivative" and "indefinite integral" are essentially the same, for arbitrary functions the two concepts do not coincide. For example, let

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0. \end{cases}$$

Then f has an indefinite integral F where

$$F(x) = \begin{cases} \int_0^x 1\ dx = x & \text{if } x > 0. \\ -\int_0^x 1\ dx = -x & \text{if } x \leq 0. \end{cases}$$

Thus $F(x) = |x|$. Then F is not an antiderivative for f, since we know that F is not differentiable at 0.

I will always try to make it clear whether $\int f$ represents an antiderivative or an indefinite integral in cases where it makes a difference.

The problem of calculating derivatives is straightforward. By using known formulas and rules, you can easily find the derivative of almost any function you can write down. The problem of calculating antiderivatives is much more complicated. In fact, none of the five functions

$$e^{x^2}, \quad \ln\Big(\ln(x)\Big), \quad \frac{1}{\ln(x)}, \quad \frac{\sin(x)}{x}, \quad \frac{(1-x)^{\frac{3}{5}}}{x^{\frac{12}{5}}} \tag{17.2}$$

have antiderivatives that can be expressed in terms of functions we have studied. (To find a proof of this assertion, see [40, page 37 ff] and [41].) The first two functions in this list are compositions of functions that have simple antiderivatives, the third function is the reciprocal of a function with a simple antiderivative, and each of the last two functions is a product of two functions with simple antiderivatives. (An antiderivative for ln will be calculated in (17.25).) It follows that there is no chain rule or reciprocal rule or product rule for calculating antiderivatives. We will see, however, that the chain rule and the product rule for differentiation do give rise to antidifferentiation formulas.

The five functions

$$e^{\sqrt{x}}, \quad \sin\Big(\ln(x)\Big), \quad \frac{1}{\sin x}, \quad \frac{\ln(x)}{x}, \quad \frac{(1-x)^{\frac{2}{5}}}{x^{\frac{12}{5}}}, \tag{17.3}$$

which look somewhat similar to the functions in (17.2), turn out to have simple antiderivatives, as you will see in (17.42c), (17.22), (17.7), (17.31f) and (17.41). It is often not easy to tell the difference between a function that has a simple antiderivative and a function that does not.

Many simple functions that arise in applied problems do not have simple antiderivatives. The exercises in this chapter have been carefully designed to be non-typical functions whose antiderivatives can be found.

The Maple instructions for finding antiderivatives and integrals are

$$\texttt{int(f(x) , x);} \quad = \int f(x)dx,$$

and

$$\texttt{int(f(x) , x = a..b);} \quad = \int_a^b f(x)dx.$$

I gave the five functions in (17.2) to Maple to antidifferentiate.

The results were:

```
> int( exp(x^2),x);
```

$$-\frac{1}{2}\,I\,\sqrt{\pi}\,\mathrm{erf}(\,I\,x\,)$$

```
> int( ln(ln(x)),x);
```

$$x\ln(\,\ln(\,x\,)\,)+\mathrm{Ei}(\,1,-\ln(\,x\,)\,)$$

```
> int(1/ln(x),x);
```

$$-\mathrm{Ei}(\,1,-\ln(\,x\,)\,)$$

```
> int(sin(x)/x,x);
```

$$\mathrm{Si}(\,x\,)$$

```
> int( ((1-x)^(3/5))/ (x^(12/5)),x);
```

$$-\frac{5}{14}\,\frac{2-5\,x+3\,x^2}{x\sqrt[5]{(-1+x)^2\,x^2}}-\int -3/14\,\frac{1}{\sqrt[5]{(-1+x)^2\,x^2}}dx$$

In the first four cases, an answer has been given involving names of functions we have not seen before, (and which we will not see again in this course). The definitions of these functions are:

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt, \tag{17.4}$$

$$\mathrm{Ei}(n,x) = \lim_{N\to\infty}\int_1^N \frac{e^{-xt}}{t^n}\,dt, \qquad (n\in\mathbf{Z}^+) \tag{17.5}$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt. \tag{17.6}$$

In equation (17.4), we assume that $\frac{\sin(t)}{t}=1$ when $t=0$. The function Si is called the *sine integral*. In equation (17.5), $\mathrm{Ei}(n,x)$ makes sense only when x is positive. The definition of $\mathrm{Ei}(n,x)$ for $x<0$ involves ideas we have not discussed. The function Ei is called the *exponential integral*.

The function erf is called the *error function*. The answer given by Maple for $\int e^{x^2}\,dx$ involves the symbol I. This is Maple's notation for $\sqrt{-1}$. The definition of $\mathrm{erf}(Ix)$ makes no sense in terms of concepts we have studied. However you can use Maple to calculate integrals even if you do not know what the symbols mean. The following instructions find $\int_0^1 \exp(x^2)dx$:

```
> int( exp(x^2),x= 0..1);
```

$$-\frac{1}{2} I \operatorname{erf}(I)\sqrt{\pi}$$

```
> evalf(%);
```

$$1.462651746$$

17.2 Basic Formulas

Every differentiation formula gives rise to an antidifferentiation formula. We review here a list of formulas that you should know. In each case you should verify the formula by differentiating the right side. You should know these formulas backward and forward.

$$\begin{aligned}
\int (f(x))^r f'(x)dx &= \frac{(f(x))^{r+1}}{r+1} \quad (r \neq -1). \\
\int \frac{f'(x)}{f(x)}dx &= \ln(|f(x)|). \\
\int \cos(f(x))f'(x)dx &= \sin(f(x)). \\
\int \sin(f(x))f'(x)dx &= -\cos(f(x)). \\
\int e^{f(x)}f'(x)dx &= e^{f(x)}. \\
\int \sec^2(f(x))f'(x)dx &= \tan(f(x)). \\
\int \csc^2(f(x))f'(x)dx &= -\cot(f(x)). \\
\int \sec(f(x))\tan(f(x))f'(x)dx &= \sec(f(x)). \\
\int \csc(f(x))\cot(f(x))f'(x)dx &= -\csc(f(x)). \\
\int \frac{f'(x)}{1+f^2(x)}dx &= \arctan(f(x)). \\
\int \frac{f'(x)}{\sqrt{1-f^2(x)}}dx &= \arcsin(f(x)).
\end{aligned}$$

17.7 Exercise. Verify that

$$\frac{d}{dx}\Big(\ln(|\sec(f(x))+\tan(f(x))|)\Big) = \sec(f(x))f'(x)$$

and

$$\frac{d}{dx}\Big(\ln(|\csc(f(x))+\cot(f(x))|)\Big)=-\csc(f(x))f'(x).$$

It follows from the previous exercise that

$$\int \sec(f(x))f'(x)dx=\ln(|\sec(f(x))+\tan(f(x))|)$$

and

$$\int \csc(f(x))f'(x)dx=-\ln(|\csc(f(x))+\cot(f(x))|).$$

You should add these two formulas to the list of antiderivatives to be memorized.

17.8 Theorem (Sum rule for antiderivatives) *If f and g are functions that have antiderivatives on some interval $[a,b]$, and if $c\in\mathbf{R}$ then $f+g$, $f-g$ and cf have antiderivatives on $[a,b]$ and*

$$\int(f\pm g)=\int f\pm\int g,$$

and

$$\int cf=c\int f. \tag{17.9}$$

Proof: The meaning of this statement is that if F is an antiderivative for f and G is an antiderivative for G, then $F\pm G$ is an antiderivative for $f\pm g$, and cF is an antiderivative for cf. The warning about the ambiguous notation for indefinite integrals given on page 214 applies also to antiderivatives.

Let F,G be antiderivatives for f and g respectively on $[a,b]$. Then F and G are continuous on $[a,b]$, and

$$F'=f \text{ and } G'=g$$

on (a,b). Hence $F\pm G$ are continuous on $[a,b]$, and

$$(F\pm G)'=F'\pm G'=f\pm g$$

on (a,b), and hence

$$\int(f\pm g)=F\pm G=\int f\pm\int g.$$

Also cF is continuous on $[a, b]$, and

$$(cF)' = cF' = cf$$

on (a, b), so that

$$\int cf = cF = c\int f. \parallel$$

17.10 Example. We will calculate $\int x^2(x^3+1)^3 dx$.
I will try to bring this integral into the form

$$\int (f(x))^r f'(x) dx.$$

It appears reasonable to take $f(x) = (x^3+1)$, and then $f'(x) = 3x^2$. The $3x^2$ doesn't quite appear in the integral, but I can get it where I need it by multiplying by a constant, and using (17.9):

$$\int x^2(x^3+1)^3 dx = \frac{1}{3}\int (3x^2)(x^3+1)^3 dx = \frac{1}{3}\frac{(x^3+1)^4}{4} = \frac{(x^3+1)^4}{12}.$$

17.11 Example. We will calculate $\int x(x^3+1)^3 dx$.
This problem is more complicated than the last one. Here I still want to take $f(x) = (x^3+1)$, but I cannot get the "$f'(x)$" that I need. I will multiply out $(x^3+1)^3$

$$\begin{aligned}
\int x(x^3+1)^3 dx &= \int x((x^3)^3 + 3(x^3)^2 + 3(x^3) + 1) dx \\
&= \int (x^{10} + 3x^7 + 3x^4 + x) dx \\
&= \frac{x^{11}}{11} + 3\frac{x^8}{8} + 3\frac{x^5}{5} + \frac{x^2}{2}.
\end{aligned}$$

17.12 Example. We will calculate $\int te^{t^2} dt$.

$$\int te^{t^2} dt = \frac{1}{2}\int (2t)e^{t^2} dt.$$

Since $\dfrac{d}{dt}(t^2) = 2t$ we get

$$\int te^{t^2} dt = \frac{1}{2}e^{t^2}.$$

17.13 Example. We will consider $\int e^{t^2} dt$.
Although this problem looks similar to the one we just did, it can be shown that no function built up from the functions we have studied by algebraic operations is an antiderivative for $\exp(t^2)$. So we will not find the desired antiderivative. (But by the fundamental theorem of the calculus we know that e^{t^2} has an antiderivative.)

17.14 Example. We will calculate $\int \tan$.

$$\int \tan = \int \frac{\sin}{\cos} = -\int \frac{\cos'}{\cos} = -\ln(|\cos|).$$

17.15 Example. We will calculate $\int_0^a \frac{1}{a^2+x^2}\, dx$.
The integrand $\frac{1}{a^2+x^2}$ looks enough like $\frac{1}{1+x^2}$ that I will try to get an arctan from this integral.

$$\int \frac{1}{a^2+x^2} = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\, dx$$

Now $\frac{d}{dx}\left(\frac{x}{a}\right) = \frac{1}{a}$, so

$$\int \frac{1}{a^2+x^2} = \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\frac{d}{dx}\left(\frac{x}{a}\right) dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right).$$

Thus we have found an antiderivative for $\frac{1}{a^2+x^2}$. Hence

$$\int_0^a \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\arctan(\frac{x}{a})\Big|_0^a = \frac{1}{a}\arctan(1) = \frac{\pi}{4a}.$$

17.16 Exercise. Find the following antiderivatives:

a) $\int e^x \sin(e^x) dx$.

b) $\int \frac{e^x}{\sin(e^x)} dx$.

c) $\int (3w^4 + w)^2 (12w^3 + 1) dw$.

d) $\int \cos(4x) dx$.

e) $\int \frac{2x}{1+x^2} dx$.

f) $\int \cot(2x) dx$.

g) $\int \frac{2}{1+w^2} dw$.

h) $\int \frac{2w}{1+w^2} dw$.

i) $\int \sin^3(x) dx$.

j) $\int \sin^4(x) dx$.

17.3 Integration by Parts

17.17 Theorem (Integration by parts.) *Let f, g be functions that are continuous on an interval $[a, b]$ and differentiable on (a, b). If $f'g$ has an antiderivative on $[a, b]$, then $g'f$ also has an antiderivative on $[a, b]$ and*

$$\int g'f = fg - \int f'g. \tag{17.18}$$

We call formula (17.18) the formula for integration by parts.

Proof: This theorem is just a restatement of the product rule for differentiation. If f and g are differentiable on (a, b) then the product rule says that

$$(fg)' = f'g + g'f$$

so that

$$g'f = (fg)' - f'g$$

on (a,b). If $\int f'g$ is an antiderivative for $f'g$ on $[a,b]$, then $fg - \int f'g$ is continuous on $[a,b]$, and

$$(fg - \int f'g)' = (fg)' - f'g = g'f$$

on (a,b). We have shown that $fg - \int f'g$ is an antiderivative for $g'f$ on $[a,b]$. ‖

17.19 Example. We will calculate $\int_0^\pi x\sin(3x)dx$. We begin by searching for an antiderivative for $x\sin(3x)$. Let

$$\begin{aligned} f(x) &= x, \\ g'(x) &= \sin(3x), \\ f'(x) &= 1, \\ g(x) &= -\frac{1}{3}\cos(3x). \end{aligned}$$

Then by the formula for integration by parts

$$\begin{aligned} \int x\sin(3x)dx &= \int f(x)g'(x)dx \\ &= f(x)g(x) - \int f'(x)g(x)dx \\ &= -\frac{x}{3}\cos(3x) + \frac{1}{3}\int \cos(3x)dx \\ &= -\frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x). \end{aligned} \tag{17.20}$$

Hence

$$\begin{aligned} \int_0^\pi x\sin(3x)dx &= \left(-\frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x)\right)\Big|_0^\pi \\ &= -\frac{\pi}{3}\cos(3\pi) = \frac{\pi}{3}. \end{aligned}$$

Suppose I had tried to find $\int x\sin(3x)$ in the following way: Let

$$\begin{aligned} f(x) &= \sin(3x), \\ g'(x) &= x, \\ f'(x) &= 3\cos(3x), \\ g(x) &= \frac{1}{2}x^2. \end{aligned}$$

Then by the formula for integration by parts

$$\begin{aligned}\int x\sin(3x)dx &= \int f(x)g'(x)dx \\ &= f(x)g(x) - \int f'(x)g(x)dx \\ &= \frac{1}{2}x^2\sin(3x) - \frac{3}{2}\int x^2\cos(3x)dx. \qquad (17.21)\end{aligned}$$

In this case the antiderivative $\int x^2\cos(3x)dx$ looks more complicated than the one I started out with. When you use integration by parts, it is not always clear what you should take for f and for g'. If you find things are starting to look more complicated rather than less complicated, you might try another choice for f and g'.

Integration by parts is used to evaluate antiderivatives of the forms $\displaystyle\int x^n\sin(ax)dx$, $\displaystyle\int x^n\cos(ax)dx$, and $\displaystyle\int x^n e^x dx$ when n is a positive integer. These can be reduced to antiderivatives of the forms $\displaystyle\int x^{n-1}\sin(ax)dx$, $\displaystyle\int x^{n-1}\cos(ax)dx$, and $\displaystyle\int x^{n-1}e^x dx$, so by applying the process n times we get the power of x down to x^0, which gives us antiderivatives we can easily find.

17.22 Example. We will calculate $\int\sin(\ln(x))$.
Let

$$\begin{aligned}f(x) &= \sin(\ln(x)), \\ g'(x) &= 1, \\ g(x) &= x, \\ f'(x) &= \frac{\cos(\ln(x))}{x}.\end{aligned}$$

Then

$$\begin{aligned}\int\sin(\ln(x))dx &= \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \\ &= x\sin(\ln(x)) - \int\cos(\ln(x))dx \qquad (17.23)\end{aligned}$$

We will now use integration by parts to find an antiderivative for $\cos(\ln(x))$.
Let

$$F(x) = \cos(\ln(x)),$$

$$\begin{aligned} G'(x) &= 1, \\ G(x) &= x, \\ F'(x) &= -\frac{\sin(\ln(x))}{x}. \end{aligned}$$

Then

$$\begin{aligned} \int \cos(\ln(x))dx &= \int F(x)G'(x) = F(x)G(x) - \int F'(x)G(x)dx \\ &= x\cos(\ln(x)) + \int \sin(\ln(x))dx \end{aligned} \tag{17.24}$$

From equations (17.23) and (17.24) we see that

$$\int \sin(\ln(x))dx = x\sin(\ln(x)) - \Big(x\cos(\ln(x)) + \int \sin(\ln(x))\Big).$$

Thus

$$2\int \sin(\ln(x))dx = x\sin(\ln(x)) - x\cos(\ln(x)),$$

and

$$\int \sin(\ln(x))dx = \frac{x}{2}(\sin(\ln(x)) - \cos(\ln(x))).$$

17.25 Example. We will calculate $\int \ln(t)dt$. Let

$$\begin{aligned} f(t) &= \ln(t), \\ g'(t) &= 1, \\ g(t) &= t, \\ f'(t) &= \frac{1}{t}. \end{aligned}$$

Then

$$\begin{aligned} \int \ln(t)dt &= \int f(t)g'(t)dt = f(t)g(t) - \int f'(t)g(t)dt \\ &= t\ln(t) - \int 1dt \\ &= t\ln(t) - t. \end{aligned}$$

17.26 Theorem (Antiderivative of inverse functions.) *Let I and J be intervals in $\mathbf{R}$, and let $f\colon I \to J$ be a continuous function such that $f'(x)$ is defined and non-zero for all x in the interior of I. Suppose that $g\colon J \to I$ is* *the inverse function for f, and that F is an antiderivative for f. Then an antiderivative for g on J is given by*

$$\int g(x)\,dx = xg(x) - (F\circ g)(x). \tag{17.27}$$

Proof: Let $h(x) = x$. Then $h'(x) = 1$, and

$$\begin{aligned}\int g(x)\,dx &= \int g(x)h'(x)dx = g(x)h(x) - \int g'(x)h(x)dx \\ &= xg(x) - \int xg'(x)dx \end{aligned} \tag{17.28}$$

Now $F' = f$ and $f\circ g(x) = x$ for all x in J, so

$$(F\circ g)'(x) = F'\big(g(x)\big)\cdot g'(x) = f\big(g(x)\big)\cdot g'(x) = xg'(x)$$

and it follows from (17.28) that

$$\begin{aligned}\int g(x)\,dx &= xg(x) - \int (F\circ g)'(x)\,dx \\ &= xg(x) - (F\circ g)(x). \ \|\end{aligned}$$

Remark: It follows from the proof of the previous theorem that if you know an antiderivative for a function f, then you can find an antiderivative for the inverse function g by integration by parts. This is what you should remember about the theorem. The formula (17.27) is not very memorable.

17.29 Example. In the previous theorem, if we take

$$f(x) = e^x, \quad F(x) = e^x, \quad g(x) = \ln(x),$$

then we get

$$\int \ln(x)\,dx = x\ln(x) - e^{\ln(x)} = x\ln(x) - x.$$

This agrees with the result obtained in example 17.25.

17.30 Exercise. What is wrong with the following argument? Let

$$\begin{aligned} f(x) &= \frac{1}{x}, \\ g'(x) &= 1, \\ f'(x) &= -\frac{1}{x^2}, \\ g(x) &= x. \end{aligned}$$

Then

$$\begin{aligned}\int \frac{1}{x}dx &= \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \\ &= 1 + \int \frac{1}{x}dx.\end{aligned}$$

If we subtract $\int \frac{1}{x}dx$ from both sides we obtain

$$0 = 1.$$

17.31 Exercise. Calculate the following antiderivatives:

a) $\int xe^x dx$.

b) $\int e^x \sin(x)dx$. (Integrate by parts more than once.)

c) $\int \arctan(u)du$.

d) $\int \frac{x}{\sqrt{4-x^2}}dx$.

e) $\int x\sqrt{4-x^2}dx$.

f) $\int x^r \ln(|x|)dx$, where $r \in \mathbf{R}$. Have you considered the case where $r = -1$?

g) $\int x^2 \cos(2x)dx$.

17.4 Integration by Substitution

We will now use the chain rule to find some antiderivatives. Let g be a real valued function that is continuous on some interval J and differentiable on the interior of J, and let f be a function such that f has an antiderivative F on some interval K. We will suppose that $g(J) \subset K$ and $g\Big(\text{interior}(J)\Big) \subset \text{interior}(K)$. It then follows that $F \circ g$ is continuous on J and differentiable on interior(J), and

$$(F \circ g)'(t) = (F' \circ g)(t)g'(t) = (f \circ g)(t)g'(t) \tag{17.32}$$

for all t in the interior of J; i.e., $F \circ g$ is an antiderivative for $(f \circ g)g'$ on J. Thus

$$\int f\big(g(t)\big)g'(t)dt = F\big(g(t)\big) \text{ where } F(u) = \int f(u)du. \tag{17.33}$$

There is a standard ritual for using (17.33) to find $\displaystyle\int f\big(g(t)\big)g'(t)dt$ when an antiderivative F can be found for f. We write:

Let $u = g(t)$. Then $du = g'(t)dt$ (or $du = \dfrac{du}{dt}dt$), so

$$\int f\big(g(t)\big)g'(t)dt = \int f(u)du = F(u) = F\big(g(t)\big). \tag{17.34}$$

In the first equality of (17.34) we replace $g(t)$ by u and $g'(t)dt$ by du, and in the last step we replace u by $g(t)$. Since we have never assigned any meaning to "*du*" or "*dt*", we should think of (17.34) just as a mnemonic device for remembering (17.33).

17.35 Example. Find $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}dx$.

Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}dx$, so

$$\begin{aligned} \int \frac{\sin(\sqrt{x})}{\sqrt{x}}dx &= 2\int \sin(\sqrt{x})\frac{1}{2\sqrt{x}}dx \\ &= 2\int \sin(u)du = -2\cos(u) \\ &= -2\cos(\sqrt{x}). \ \| \end{aligned}$$

Suppose we want to find $\int \sin(\sqrt{x})dx$. If we had a $\sqrt{x}$ in the denominator, this would be a simple problem. (In fact we just considered this problem in the previous example.) We will now discuss a method of introducing the missing $\sqrt{x}$.

Suppose g is a function on an interval J such that $g'(t)$ is never zero on the interior of J, and suppose that h is an inverse function for g. Then

$$\Big(h\big(g(x)\big) = x\Big) \implies \Big(h'\big(g(x)\big) \cdot g'(x) = 1\Big)$$

for all x in the interior of J, so

$$\int f\Big(g(x)\Big)dx = \int f\Big(g(x)\Big) \cdot h'\Big(g(x)\Big) \cdot g'(x)dx.$$

We now apply the ritual (17.34): Let $u = g(x)$. Then $du = g'(x)dx$, so

$$\begin{aligned}\int f\Big(g(x)\Big)dx &= \int f\Big(g(x)\Big)h'\Big(g(x)\Big) \cdot g'(x)dx \\ &= \int f(u)h'(u)du.\end{aligned}$$

If we can find an antiderivative H for fh', then

$$\int f(u)h'(u)du = H(u) = H\Big(g(x)\Big).$$

We have shown that if h is an inverse function for g, then

$$\int f\Big(g(x)\Big)dx = H\Big(g(x)\Big) \text{ where } H(u) = \int f(u)h'(u)du \tag{17.36}$$

There is a ritual associated with this result also. To find $\int f\Big(g(x)\Big)dx$:
Let $u = g(x)$. Then $x = h(u)$ so $dx = h'(u)du$.
Hence

$$\int f\Big(g(x)\Big)dx = \int f(u)h'(u)du = H(u) = H\Big(g(x)\Big). \tag{17.37}$$

17.38 Example. To find $\int \sin(\sqrt{x})dx$.
Let $u = \sqrt{x}$. Then $x = u^2$ so $dx = 2u\,du$.
Thus

$$\int \sin(\sqrt{x})dx = \int \sin(u) \cdot 2u\,du = 2\int u\sin(u)du.$$

We can now use integration by parts to find $\int u\sin(u)du$. Let

$$\begin{aligned} f(u) = u, \qquad g'(u) &= \sin(u), \\ f'(u) = 1, \qquad g(u) &= -\cos(u). \end{aligned}$$

Then

$$\begin{aligned}
\int u\sin(u)du &= \int f(u)g'(u)du \\
&= f(u)g(u) - \int f'(u)g(u)du \\
&= -u\cos(u) + \int \cos(u)du \\
&= -u\cos(u) + \sin(u).
\end{aligned}$$

Hence

$$\begin{aligned}
\int \sin(\sqrt{x})dx &= 2\int u\sin u\ du \\
&= -2u\cos(u) + 2\sin(u) \\
&= -2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}).
\end{aligned}$$

17.39 Example. To find $\int \frac{1}{e^x+e^{-x}}dx$.

Let $u = e^x$. Then $x = \ln(u)$ so $dx = \frac{1}{u}du$.

$$\begin{aligned}
\int \frac{1}{e^x+e^{-x}}dx &= \int \frac{1}{(u+\frac{1}{u})}\cdot\frac{1}{u}du = \int \frac{1}{u^2+1}du \\
&= \arctan(u) = \arctan(e^x).
\end{aligned}$$

17.40 Example. To find $\int t\sqrt{t+1}\ dt$.

Let $u = t+1$. Then $t = u-1$ so $dt = du$.

Hence

$$\begin{aligned}
\int t\sqrt{t+1}\ dt &= \int (u-1)\sqrt{u}\ du = \int u^{3/2} - u^{1/2}du \\
&= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} = \frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2}.
\end{aligned}$$

17.41 Example. To find $\int \frac{(1-x)^{\frac{2}{5}}}{x^{\frac{12}{5}}} dx$.

$$\int \frac{(1-x)^{\frac{2}{5}}}{x^{\frac{12}{5}}} dx = \int \left(\frac{1-x}{x}\right)^{\frac{2}{5}} \cdot \frac{1}{x^2} dx.$$

Let $u = \frac{1-x}{x} = \frac{1}{x} - 1$. Then $du = -\frac{1}{x^2} dx$, and

$$\int \frac{(1-x)^{\frac{2}{5}}}{x^{\frac{12}{5}}} dx = -\int u^{\frac{2}{5}} du = -\frac{5}{7} u^{\frac{7}{5}} = -\frac{5}{7} \left(\frac{1-x}{x}\right)^{\frac{7}{5}}.$$

Thus

$$\int \frac{(1-x)^{\frac{2}{5}}}{x^{\frac{12}{5}}} dx = -\frac{5}{7} \left(\frac{1-x}{x}\right)^{\frac{7}{5}}.$$

17.42 Exercise. Find the following antiderivatives:

a) $\int x^2 \sin(x^3) dx$.

b) $\int \frac{e^x}{1+e^x} dx$.

c) $\int e^{\sqrt{x}} dx$.

d) $\int \frac{\ln(3x)}{x} dx$.

e) $\int 2^x \, dx$.

f) $\int \frac{e^{2x} + e^{3x}}{e^{4x}} \, dx$.

g) $\int x(1 + \sqrt[3]{x}) dx$.

17.5 Trigonometric Substitution

Integrals of the form $\int F(\sqrt{a^2+x^2})dx$ and $\int F(\sqrt{a^2-x^2})dx$ often arise in applications. There is a special trick for dealing with such integrals. Since

$$x = a\tan(\arctan(\frac{x}{a})) \text{ for all } x \in \mathbf{R},$$

we can write

$$\int F(\sqrt{a^2+x^2})dx = \int F(\sqrt{a^2 + \Big(a\tan(\arctan(\frac{x}{a}))\Big)^2})dx.$$

If we now make the substitution

$$u = \arctan(\frac{x}{a}) \text{ or } x = a\tan(u), \qquad \Big(u \in (-\frac{\pi}{2}, \frac{\pi}{2})\Big)$$

then we find $dx = a\sec^2(u)du$, or

$$\int F(\sqrt{a^2+x^2})dx = \int F(\sqrt{a^2 + \Big(a\tan(u)\Big)^2})a\sec^2 u\ du.$$

Now

$$a^2 + \Big(a\tan(u)\Big)^2 = a^2\Big(1+\tan^2(u)\Big) = a^2\sec^2(u)$$

so

$$\sqrt{a^2 + \Big(a\tan(u)\Big)^2} = a\sec(u).$$

(Since $u \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)$ we have $\sec(u) > 0$ and the square root is positive.) Thus

$$\int F(\sqrt{a^2+x^2})dx = a\int F\Big(a\sec(u)\Big)\cdot\sec^2(u)du.$$

Often this last antiderivative can be found. If

$$a\int F\Big(a\sec(u)\Big)\cdot\sec^2(u)du = H(u),$$

then by the ritual (17.37)

$$\int F(\sqrt{a^2+x^2})dx = a\int F\Big(a\sec(u)\Big)\cdot\sec^2(u)du = H(u) = H(\arctan(\frac{x}{a})).$$

The ritual to apply when using this method for finding $\int F(\sqrt{a^2+x^2})dx$ is:

Let $x = a\tan(u)$. Then $dx = a\sec^2(u)du$, and

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2(u)} = \sqrt{a^2\sec^2(u)} = a\sec(u),$$

so

$$\int F(\sqrt{a^2+x^2})dx = a\int F\Big(a\sec(u)\Big)\sec^2(u)du = H(u) = H(\arctan(\frac{x}{a})).$$

There is a similar ritual for integrals of the form $\int F(\sqrt{a^2-x^2})dx$ (Here we will just describe the ritual).

Let $x = a\sin(u)$. Then $dx = a\cos(u)du$ and

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin(u)} = \sqrt{a^2\cos^2(u)} = a\cos(u) \tag{17.43}$$

so

$$\int F(\sqrt{a^2-x^2})dx = a\int F\Big(a\cos(u)\Big)\cdot\cos(u)du = H(u) = H(\arcsin(\frac{x}{a})).$$

Observe that in equation (17.43) we are assuming that $u = \arcsin(\frac{x}{a})$, so $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so $\cos(u) \geq 0$, and the sign of the square root is correct.

17.44 Example. Find $\int\sqrt{4+x^2}dx$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$, and

$$\sqrt{4+x^2} = \sqrt{4(1+\tan^2\theta)} = 2\sqrt{\sec^2(\theta)} = 2\sec(\theta). \tag{17.45}$$

Thus

$$\int\sqrt{4+x^2}\,dx = 2^2\int\sec\theta\cdot\sec^2\theta\,d\theta = 4\int\sec^3(\theta)\,d\theta.$$

To find $\int\sec^3(\theta)d\theta$, I will integrate by parts. Let

$$\begin{array}{ll} f(\theta) = \sec(\theta), & g'(\theta) = \sec^2(\theta), \\ f'(\theta) = \sec(\theta)\tan(\theta), & g(\theta) = \tan(\theta). \end{array}$$

Hence,

$$\begin{aligned}
\int \sec^3(\theta)d\theta &= \int f(\theta)g'(\theta)d\theta \\
&= f(\theta)g(\theta) - \int f'(\theta)g(\theta)d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec(\theta)\tan^2(\theta)d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec(\theta)(\sec^2(\theta)-1)d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec^3(\theta)d\theta + \int \sec(\theta)d\theta.
\end{aligned}$$

Hence

$$\begin{aligned}
2\int \sec^3(\theta)d\theta &= \sec(\theta)\tan(\theta) + \int \sec(\theta)d\theta \\
&= \sec(\theta)\tan(\theta) + \ln(|\sec(\theta)+\tan(\theta)|);
\end{aligned}$$

i.e.,

$$\int \sec^3(\theta)d\theta = \frac{1}{2}\Bigg(\sec(\theta)\tan(\theta) + \ln\Big(|\sec(\theta)+\tan(\theta)|\Big)\Bigg). \tag{17.46}$$

Hence

$$\begin{aligned}
\int \sqrt{4+x^2}dx &= 4\int \sec^3(\theta)d\theta \\
&= 2\Bigg(\sec(\theta)\tan(\theta) + \ln\Big(|\sec(\theta)+\tan(\theta)|\Big)\Bigg).
\end{aligned}$$

By (17.45) we have $\tan(\theta) = \frac{x}{2}$ and $\sec(\theta) = \frac{1}{2}\sqrt{4+x^2}$. Thus

$$\begin{aligned}
\int \sqrt{4+x^2}dx &= 2\Bigg(\frac{1}{2}\sqrt{4+x^2}\cdot\frac{x}{2} + \ln\Big(\Big|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\Big|\Big)\Bigg) \\
&= \frac{x\sqrt{4+x^2}}{2} + 2\ln\Big(\Big|\frac{\sqrt{4+x^2}+x}{2}\Big|\Big).
\end{aligned}$$

17.47 Example. In the process of working out the last example we found $\int \sec^3(\theta)d\theta$ using integration by parts. Here is a different tricky way of finding the same integral [32].

$$\begin{aligned}
\int \sec^3(\theta)d\theta &= \frac{1}{2}\int(\sec^3(\theta) + \sec^3(\theta))d\theta \\
&= \frac{1}{2}\int(\sec(\theta)(1+\tan^2(\theta)) + \sec^3(\theta))d\theta \\
&= \frac{1}{2}\int(\sec(\theta) + \Big((\sec(\theta)\tan(\theta))\cdot\tan(\theta) + \sec(\theta)\cdot\sec^2(\theta)\Big))d\theta \\
&= \frac{1}{2}\int(\sec(\theta) + \frac{d}{d\theta}(\sec(\theta)\tan(\theta)))d\theta \\
&= \frac{1}{2}(\ln(|\sec(\theta)+\tan(\theta)|) + \sec(\theta)\tan(\theta)).
\end{aligned}$$

17.48 Example. Find $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}dx.$

Let $x = a\sin(\theta)$. Then $dx = a\cos(\theta)d\theta$ and

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\sqrt{\cos^2\theta} = a\cos\theta.$$

Thus

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2-x^2}}dx &= \int \frac{a\cos(\theta)}{a\cos(\theta)}d\theta = \int 1\ d\theta \\
&= \theta = \arcsin(\frac{x}{a}).
\end{aligned}$$

17.49 Exercise. Find the following antiderivatives:

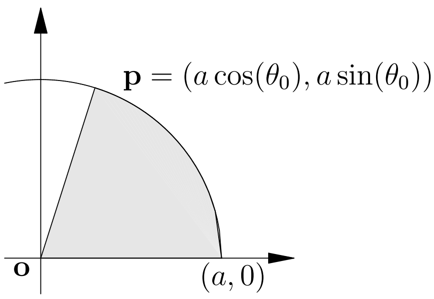
a) $\displaystyle\int \sqrt{a^2-x^2}\ dx$

b) $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}}\ dx$

c) $\displaystyle\int \frac{x}{\sqrt{a^2-x^2}}\ dx$

d) $\displaystyle\int x\sqrt{a^2+x^2}\ dx$

17.50 Example (Area of a circular sector) Let a be a positive number, and let θ_0 be a number in $[0, \frac{\pi}{2})$. Let $\mathbf{o} = (0,0)$, and let $\mathbf{p} = (a\cos(\theta_0), a\sin(\theta_0))$. Let $T(a, \theta_0)$ be the circular sector bounded by the positive x-axis, the segment $[\mathbf{op}]$, and the circle $\{x^2 + y^2 = a^2\}$.



$T(a, \theta_0)$ is shaded region

The equation for $[\mathbf{op}]$ is

$$y = \frac{a\sin(\theta_0)}{a\cos(\theta_0)}x = x\tan(\theta_0),$$

and the equation for the upper semicircle is

$$y = \sqrt{a^2 - x^2}.$$

Hence

$$\text{area}(T(a, \theta_0)) = A_0^a(f),$$

where

$$f(x) = \begin{cases} x\tan(\theta_0) & \text{if } 0 \leq x \leq a\cos(\theta_0), \\ \sqrt{a^2 - x^2} & \text{if } a\cos(\theta_0) \leq x \leq a. \end{cases}$$

i.e.

$$\text{area}(T(a, \theta_0)) = \int_0^{a\cos(\theta_0)} x\tan(\theta_0)dx + \int_{a\cos(\theta_0)}^a \sqrt{a^2 - x^2}\, dx.$$

In exercise 17.49.a you showed that

$$\int \sqrt{a^2 - x^2} = \frac{1}{2}a^2 \arcsin\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2 - x^2},$$

so

$$
\begin{aligned}
\text{area}(T(a,\theta_0)) &= \tan(\theta_0)\left.\frac{x^2}{2}\right|_0^{a\cos(\theta_0)} + \left(\frac{1}{2}a^2\arcsin(\frac{x}{a}) + \frac{1}{2}x\sqrt{a^2-x^2}\right)\Bigg|_{a\cos(\theta_0)}^{a} \\
&= \frac{1}{2}\tan(\theta_0)a^2\cos^2(\theta_0) + \frac{1}{2}a^2\arcsin(1) \\
&\qquad -\frac{1}{2}a^2\arcsin(\cos(\theta_0)) - \frac{1}{2}a\cos(\theta_0)\sqrt{a^2 - a^2\cos^2(\theta_0)} \\
&= \frac{a^2}{2}\sin(\theta_0)\cos(\theta_0) + \frac{\pi a^2}{4} \\
&\qquad -\frac{a^2}{2}\arcsin(\sin(\frac{\pi}{2}-\theta_0)) - \frac{a^2}{2}\cos(\theta_0)\sqrt{1-\cos^2(\theta_0)} \\
&= \frac{a^2}{2}\sin(\theta_0)\cos(\theta_0) + \frac{\pi a^2}{4} - \frac{a^2}{2}(\frac{\pi}{2}-\theta_0) - \frac{a^2}{2}\sin(\theta_0)\cos(\theta_0) \\
&= \frac{1}{2}a^2\theta_0.
\end{aligned}
$$

By using symmetry arguments, you can show that this formula actually holds for $0 \le \theta_0 \le 2\pi$.

17.6 Substitution in Integrals

Let f be a nice function on an interval $[a,b]$. Then if F is any antiderivative for f, we have

$$\int_a^b f = F\,|_a^b = F(b) - F(a),$$

by the fundamental theorem of calculus. We saw in (17.32) that under suitable hypotheses on g, $F \circ g$ is an antiderivative for $(f \circ g)g'$. Hence

$$\int_a^b f\Big(g(t)\Big)g'(t)dt = F \circ g\,|_a^b = F\Big|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)du.$$

Hence we can find $\displaystyle\int_a^b f\Big(g(t)\Big)g'(t)dt$ by the following ritual:

Let $u = g(t)$. When $t = a$ then $u = g(a)$ and when $t = b$ then $u = g(b)$. Also $du = g'(t)dt$. Hence

$$\int_a^b f\Big(g(t)\Big)g'(t)dt = \int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{g(a)}^{g(b)}.$$

17.51 Example. To find $\displaystyle\int_{\pi^2}^{4\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}dx$.

Let $u = \sqrt{x}$. When $x = \pi^2$, then $u = \pi$, and when $x = 4\pi^2$, then $u = 2\pi$. Also $du = \dfrac{1}{2\sqrt{x}}dx$, so

$$\begin{aligned}\int_{\pi^2}^{4\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}dx &= 2\int_{\pi}^{2\pi} \sin(u)du = -2\cos u\Big|_{\pi}^{2\pi} \\ &= -2\Big(\cos(2\pi) - \cos(\pi)\Big) = -2(1+1) = -4.\end{aligned}$$

We saw in (17.36) that if h is an inverse function for g, then an antiderivative for $f \circ g$ is $H \circ g$, where H is an antiderivative for $f \cdot h'$. Thus

$$\int_a^b f\Big(g(t)\Big)dt = H \circ g \,|_a^b = H\Big|_{g(a)}^{g(b)}.$$

The ritual for finding $\displaystyle\int_a^b f\Big(g(t)\Big)dt$ in this case is:

Let $u = g(t)$. Then $t = h(u)$ and $dt = h'(u)du$. When $t = a$ then $u = g(a)$, and when $t = b$ then $u = g(b)$. Thus

$$\int_a^b f\Big(g(t)\Big)dt = \int_{g(a)}^{g(b)} f(u)h'(u)du = H(u)\Big|_{g(a)}^{g(b)}$$

where H is an antiderivative for fh'.

17.52 Example. To find $\displaystyle\int_0^{\ln(\sqrt{3})} \frac{1}{e^x + e^{-x}}dx$.

Let $u = e^x$. When $x = 0$ then $u = 1$, and when $x = \ln(\sqrt{3})$ then $u = \sqrt{3}$. Also $x = \ln(u)$, so $dx = \dfrac{1}{u}du$.

$$\begin{aligned}\int_0^{\ln(\sqrt{3})} \frac{1}{e^x + e^{-x}}dx &= \int_1^{\sqrt{3}} \frac{1}{(u + \frac{1}{u})} \cdot \frac{1}{u}du \\ &= \int_1^{\sqrt{3}} \frac{1}{u^2+1}du = \arctan(u)\,|_1^{\sqrt{3}} \\ &= \arctan(\sqrt{3}) - \arctan(1) \\ &= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.\end{aligned}$$

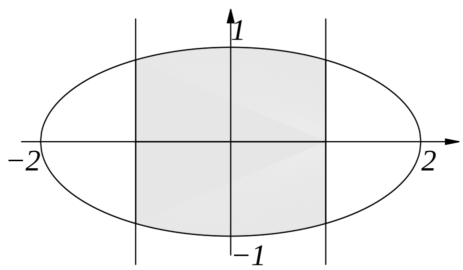
17.53 Exercise. Find the following integrals:

a) $\displaystyle\int_0^1 x^2(x^3+1)^3 dx.$

b) $\displaystyle\int_0^{3/2} \frac{1}{\sqrt{9-x^2}} dx.$

c) $\displaystyle\int_0^1 x\sqrt{1-x}\, dx.$

17.54 Exercise. Find the area of the shaded region, bounded by the ellipse $\dfrac{x^2}{4}+y^2=1$ and the lines $x=\pm 1$.



17.55 Example. In practice I would find many of the antiderivatives and integrals discussed in this chapter by computer. For example, using Maple, I would find

```
> int(sqrt(a^2+x^2),x);
```

$$\frac{1}{2}\, x\, \sqrt{a^2+x^2} + \frac{1}{2}\, a^2 \ln\left(x+\sqrt{a^2+x^2}\right)$$

```
> int(sin(sqrt(x)),x=0..Pi^2);
```

$$2\,\pi$$

```
> int(sqrt(4 - x^2),x=-1..1);
```

$$\sqrt{3}+\frac{2}{3}\,\pi$$

```
> int( (sec(x))^3,x);
```

$$\frac{1}{2}\,\frac{\sin(\,x\,)}{\cos(\,x\,)^2} + \frac{1}{2}\ln(\,\sec(\,x\,)+\tan(\,x\,)\,)$$

```
> int(exp(a*x)*cos(b*x),x);
```

$$\frac{a\,e^{(a\,x)}\cos(\,b\,x\,)}{a^2+b^2}+\frac{b\,e^{(a\,x)}\sin(\,b\,x\,)}{a^2+b^2}$$

17.7 Rational Functions

In this section we present a few rules for finding antiderivatives of simple rational functions.

To antidifferentiate $\frac{P(x)}{(x-c)^n}$ where P is a polynomial, make the substitution $u = x - c$.

17.56 Example. To find $\int \frac{x^2+1}{(x-2)^2}dx$.

Let $u = x - 2$. Then $x = 2 + u$ so $dx = du$, and

$$\begin{aligned}
\int \frac{(x^2+1)}{(x-2)^2}dx &= \int \frac{(2+u)^2+1}{u^2}du \\
&= \int \frac{u^2+4u+5}{u^2}du \\
&= \int 1+\frac{4}{u}+\frac{5}{u^2}du \\
&= u+4\ln|u|-\frac{5}{u} \\
&= (x-2)+4\ln(|x-2|)-\frac{5}{(x-2)}.
\end{aligned}$$

To find $\int \frac{R(x)}{(x-a)(x-b)}dx$ where $a \neq b$ and R is a polynomial of degree less than 2.

We will find numbers A and B such that

$$\frac{R(x)}{(x-a)(x-b)} = \frac{A}{(x-a)}+\frac{B}{(x-b)}. \tag{17.57}$$

Suppose (17.57) were valid. If we multiply both sides by $(x-a)$ we get

$$\frac{R(x)}{(x-b)} = A + \frac{B(x-a)}{x-b}.$$

Now take the limit as x goes to a to get

$$\frac{R(a)}{a-b} = A.$$

The reason I took a limit here, instead of saying "now for $x = a$ we get $\cdots$" is that a is not in the domain of the function we are considering. Similarly

$$\frac{R(x)}{x-a} = \frac{A(x-b)}{x-a} + B,$$

and if we take the limit as x goes to b, we get

$$\frac{R(b)}{b-a} = B.$$

Thus,

$$\frac{R(x)}{(x-a)(x-b)} = \frac{1}{a-b}\left[\frac{R(a)}{x-a} - \frac{R(b)}{x-b}\right]. \tag{17.58}$$

I have now shown that if there are numbers A and B such that (17.57) holds, then (17.58) holds. Since I have not shown that such numbers exist, I will verify directly that (17.58) is valid. Write $R(x) = px + q$. Then

$$\begin{aligned}
\frac{1}{a-b}\left[\frac{R(a)}{x-a} - \frac{R(b)}{x-b}\right] &= \frac{1}{a-b}\left[\frac{pa+q}{x-a} - \frac{pb+q}{x-b}\right] \\
&= \frac{1}{(a-b)}\left[\frac{(pa+q)(x-b)-(pb+q)(x-a)}{(x-a)(x-b)}\right] \\
&= \frac{1}{(a-b)}\left[\frac{x(pa-pb)-q(b-a)}{(x-a)(x-b)}\right] \\
&= \frac{1}{(a-b)}\frac{(a-b)(px+q)}{(x-a)(x-b)} \\
&= \frac{px+q}{(x-a)(x-b)} = \frac{R(x)}{(x-a)(x-b)}. \quad |||
\end{aligned}$$

17.59 Example. To find $\displaystyle\int \frac{x+1}{(x+2)(x+3)}dx$.

Let $\displaystyle\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$.

Then

$$\frac{x+1}{x+3} = A + \frac{x+2}{x+3}B,$$

so

$$A = \frac{-2+1}{-2+3} = -1,$$

and

$$\frac{x+1}{x+2} = A\frac{(x+3)}{x+2} + B,$$

so

$$B = \frac{-3+1}{-3+2} = 2.$$

Hence

$$\begin{aligned}\int \frac{x+1}{(x+2)(x+3)}dx &= \int \frac{-1}{x+2} + \frac{2}{x+3}dx \\ &= -\ln(|x+2|) + 2\ln(|x+3|).\end{aligned}$$

In this example I did not use formula (17.58), because I find it easier to remember the procedure than the general formula. I do not need to check my answer, because my proof of (17.58) shows that the procedure always works. (In practice, I usually do check the result because I am likely to make an arithmetic error.)

To find $\displaystyle\int \frac{R(x)}{x^2+ax+b}dx$ where R is a polynomial of degree < 2, and x^2+ax+b does not factor as a product of two first degree polynomials.

Complete the square to write

$$x^2+ax+b = (x-m)^2 + k.$$

Then $k > 0$, since if $k = 0$ then we have factored x^2+ax+b, and if $k < 0$ we can write $k = -n^2$, and then

$$(x-m)^2 + k = (x-m)^2 - n^2 = \Big((x-m)-n\Big)\Big((x-m)+n\Big)$$

and again we get a factorization of x^2+ax+b. Since $k > 0$, we can write $k = q^2$ for some $q \in \mathbf{R}$, and

$$x^2+ax+b = (x-m)^2 + q^2.$$

Now

$$\int \frac{R(x)}{x^2+ax+b}dx = \int \frac{R(x)}{(x-m)^2+q^2}dx.$$

Make the substitution $u = x - m$ to get an antiderivative of the form

$$\begin{aligned}\int \frac{Au+B}{u^2+q^2}du &= \frac{A}{2}\int \frac{2u}{u^2+q^2}du + B\int \frac{1}{u^2+q^2}du \\ &= \frac{A}{2}\ln(u^2+q^2) + B\int \frac{1}{u^2+q^2}du.\end{aligned}$$

The last antiderivative can be found by a trigonometric substitution.

17.60 Example. To find $\displaystyle\int \frac{u}{4u^2+8u+7}du$:

Let

$$\begin{aligned}I = \int \frac{u}{4u^2+8u+7}du &= \frac{1}{4}\int \frac{u}{u^2+2u+\frac{7}{4}}du \\ &= \frac{1}{4}\int \frac{u}{u^2+2u+1+\frac{3}{4}}du \\ &= \frac{1}{4}\int \frac{u}{(u+1)^2+\frac{3}{4}}du.\end{aligned}$$

Let $t = u+1$, so $u = t-1$ and $du = dt$. Then

$$\begin{aligned}I &= \frac{1}{4}\int \frac{t-1}{t^2+\frac{3}{4}}dt = \frac{1}{8}\int \frac{2t}{t^2+\frac{3}{4}}dt - \frac{1}{4}\int \frac{1}{t^2+\frac{3}{4}}dt \\ &= \frac{1}{8}\ln(t^2+\frac{3}{4}) - \frac{1}{4}\int \frac{1}{t^2+\frac{3}{4}}dt \\ &= \frac{1}{8}\ln\left((u+1)^2+\frac{3}{4}\right) - \frac{1}{4}\int \frac{1}{t^2+\frac{3}{4}}dt \\ &= \frac{1}{8}\ln(u^2+2u+\frac{7}{4}) - \frac{1}{4}\int \frac{1}{t^2+\frac{3}{4}}dt.\end{aligned}$$

Now let $t = \dfrac{\sqrt{3}}{2}\tan\theta$, so $dt = \dfrac{\sqrt{3}}{2}\sec^2\theta\, d\theta$, and $t^2 + \dfrac{3}{4} = \dfrac{3}{4}\sec^2\theta$. Then

$$\int \frac{1}{t^2+\frac{3}{4}}dt = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{3}{4}\sec^2\theta}d\theta = \frac{2}{\sqrt{3}}\int d\theta$$

$$
\begin{aligned}
&= \frac{2}{\sqrt{3}}\theta = \frac{2}{\sqrt{3}} \arctan\left(\frac{2t}{\sqrt{3}}\right) \\
&= \frac{2}{\sqrt{3}} \arctan\left(\frac{2u+2}{\sqrt{3}}\right).
\end{aligned}
$$

Hence,

$$I = \frac{1}{8} \ln\left(u^2 + 2u + \frac{7}{4}\right) - \frac{1}{2\sqrt{3}} \arctan\left(\frac{2u+2}{\sqrt{3}}\right).$$

To find $\displaystyle\int \frac{R(x)}{x^2+ax+b}dx$ where R is a polynomial of degree > 1.
First use long division to write

$$\frac{R(x)}{x^2+ax+b} = Q(x) + \frac{P(x)}{x^2+ax+b}$$

where Q is a polynomial, and P is a polynomial of degree ≤ 1. Then use one of the methods already discussed.

17.61 Example. To find $\displaystyle\int \frac{x^3+1}{x^2+1}dx$. By using long division, we get

$$
\begin{array}{r|rrr}
 & x & & \\
\hline
x^2+1 & x^3 & & +1 \\
 & x^3 & +x & \\
\hline
 & & -x & +1
\end{array}
$$

$$\frac{x^3+1}{x^2+1} = x + \frac{-x+1}{x^2+1}.$$

Hence

$$
\begin{aligned}
\int \frac{x^3+1}{x^2+1}dx &= \int x - \frac{x}{x^2+1} + \frac{1}{x^2+1}dx \\
&= \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + \arctan(x).
\end{aligned}
$$

17.62 Example. In exercise 17.7, you showed that $\ln\Big(|\sec(x)+\tan(x)|\Big)$ is an antiderivative for $\sec(x)$. The function $\ln\Big(|\sec(x)+\tan(x)|\Big)$ in that exercise appeared magically with no motivation. I will now derive the formula, using standard methods:

$$\int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx = \int \frac{\cos(x)}{\cos^2(x)}\,dx = \int \frac{\cos(x)}{1-\sin^2(x)}\,dx.$$

Now let $u = \sin(x)$. Then $du = \cos(x)\,dx$, and

$$\int \sec(x)\,dx = \int \frac{du}{1-u^2}.$$

Suppose $\dfrac{1}{1-u^2} = \dfrac{A}{1-u} + \dfrac{B}{1+u}$. Then

$$\frac{1}{1+u} = A + \frac{B(1-u)}{(1+u)},$$

and if we take the limit of both sides as $u \to 1$ we get $A = \dfrac{1}{2}$. Also

$$\frac{1}{1-u} = \frac{A(1+u)}{1-u} + B,$$

and if we take the limit as $u \to -1$, we get $B = \dfrac{1}{2}$. Thus

$$\begin{aligned}
\int \sec(x)\,dx &= \int \frac{1}{1-u^2}\,du \\
&= \frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right)\,du \\
&= \frac{1}{2}\left[-\ln\left(|1-u|\right) + \ln\left(|1+u|\right)\right] \\
&= \frac{1}{2}\ln\left(\left|\frac{1+u}{1-u}\right|\right) \\
&= \frac{1}{2}\ln\left(\left|\frac{1+\sin(x)}{1-\sin(x)}\right|\right).
\end{aligned}$$

Now

$$\frac{1+\sin(x)}{1-\sin(x)} = \frac{1+\sin(x)}{1-\sin(x)}\cdot\frac{1+\sin(x)}{1+\sin(x)} = \frac{\big(1+\sin(x)\big)^2}{1-\sin^2(x)} = \frac{\big(1+\sin(x)\big)^2}{\cos^2(x)},$$

so

$$\begin{aligned}
\frac{1}{2}\ln\left(\left|\frac{1+\sin(x)}{1-\sin(x)}\right|\right) &= \frac{1}{2}\ln\left(\left|\frac{1+\sin(x)}{\cos(x)}\right|^2\right) = \ln\left(\left|\frac{1+\sin(x)}{\cos(x)}\right|\right) \\
&= \ln\Big(|\sec(x)+\tan(x)|\Big),
\end{aligned}$$

and thus

$$\int \sec(x)\,dx = \ln\Big(|\sec(x)+\tan(x)|\Big).$$

17.63 Exercise. Criticize the following argument:

I want to find $\displaystyle\int \frac{x^2}{x^2-1}dx = \int \frac{x^2}{(x-1)(x+1)}dx$. Suppose

$$\frac{x^2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Then

$$\frac{x^2}{x+1} = A + \frac{(x-1)B}{x+1}.$$

If we take the limit of both sides as $x \to 1$, we get $\dfrac{1}{2} = A$. Also

$$\frac{x^2}{x-1} = \frac{A(x+1)}{x-1} + B,$$

and if we take the limit of both sides as $x \to -1$, we get $-\dfrac{1}{2} = B$. Thus

$$\frac{x^2}{x^2-1} = \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}.$$

Hence,

$$\int \frac{x^2}{x^2-1}dx = \frac{1}{2}\ln(|x-1|) - \frac{1}{2}\ln(|x+1|).$$

17.64 Exercise. Find the following antiderivatives:

a) $\displaystyle\int \frac{1}{4x^2-1}dx$

b) $\displaystyle\int \frac{1}{4x^2+1}dx$

c) $\displaystyle\int \frac{x+1}{x^2-6x+8}dx$

d) $\displaystyle\int \frac{x+1}{x^2-6x+9}dx$

e) $\displaystyle\int \frac{1}{9x^2+6x+2}dx$

f) $\displaystyle\int \frac{x^3}{x^2+1}dx$

g) $\displaystyle\int \frac{1}{\sqrt{x^2+2x+2}}dx$

17.65 Exercise. Find the following antiderivatives:

a) $\displaystyle\int \frac{\cos(ax)}{\sin^3(ax)}dx.$

b) $\displaystyle\int \frac{\sin(t)\cos(t)}{\cos^2(t)+1}dt.$

c) $\displaystyle\int \frac{1}{(1-t)^3}dt.$

d) $\displaystyle\int \frac{1}{5+4x+x^2}dx.$

e) $\displaystyle\int x^3\sqrt{x^2+1}dx.$

f) $\displaystyle\int \frac{1}{\sqrt{-3-4x-x^2}}.$

g) $\displaystyle\int \frac{\sin(2\theta)}{\cos^2(\theta)-\sin^2(\theta)}d\theta.$

h) $\displaystyle\int (1+\tan(u))^2du.$

i) Choose a number p, and find $\displaystyle\int x^p(x^{10}-2)^{10}dx.$

j) Choose a number q, and find $\displaystyle\int x^q e^{-\frac{1}{x}}dx.$

k) $\int xe^{-x^2}dx.$

l) $\int \frac{u^3}{1+u^2}du.$

m) $\int x^2 \arctan(x)dx.$

n) $\int x^3(1+x)^{\frac{1}{4}}dx.$

o) $\int xe^{2x}dx.$

p) $\int \arcsin(x)dx.$