Chapter 11

Calculation of Derivatives

11.1 Derivatives of Some Special Functions

11.1 Theorem (Derivative of power functions.) *Let $r \in \mathbf{Q}$ and let $f(x) = x^r$. Here*

$$\text{domain}(f) = \begin{cases} \mathbf{R} & \textit{if } r \in \mathbf{Z}_{\geq 0} \\ \mathbf{R} \setminus \{0\} & \textit{if } r \in \mathbf{Z}^- \\ \mathbf{R}_{\geq 0} & \textit{if } r \in \mathbf{Q}^+ \setminus \mathbf{Z} \\ \mathbf{R}^+ & \textit{if } r \in \mathbf{Q}^- \setminus \mathbf{Z}. \end{cases}$$

Let a be an interior point of domain(f). Then f is differentiable at a, and

$$f'(a) = ra^{r-1}.$$

If $r = 0$ and $a = 0$ we interpret ra^{r-1} to be 0.

Proof: First consider the case $a \neq 0$. For all x in domain$(f) \setminus \{a\}$ we have

$$\frac{f(x) - f(a)}{x - a} = \frac{x^r - a^r}{x - a} = \frac{a^r\Big((\frac{x}{a})^r - 1\Big)}{a\Big((\frac{x}{a}) - 1\Big)} = a^{r-1}\frac{\Big((\frac{x}{a})^r - 1\Big)}{(\frac{x}{a} - 1)}.$$

Let $\{x_n\}$ be a generic sequence in domain$(f) \setminus \{a\}$ such that $\{x_n\} \to a$. Let $y_n = \dfrac{x_n}{a}$. Then $\{y_n\} \to 1$ and hence by theorem 7.10 we have $\left\{\dfrac{y_n^r - 1}{y_n - 1}\right\} \to r$ and hence

$$\left\{\frac{f(x_n) - f(a)}{x_n - a}\right\} = \left\{\frac{y_n^r - 1}{y_n - 1}\right\} \cdot a^{r-1} \to ra^{r-1}.$$

This proves the theorem in the case $a \neq 0$. If $a = 0$ then $r \in \mathbf{Z}_{\geq 0}$ (since for other values of r, 0 is not an interior point of domain(f)). In this case

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^r - 0^r}{x} = \begin{cases} 0 & \text{if } r = 0 \text{ (remember } 0^0 = 1). \\ x^{r-1} & \text{if } r \neq 0. \end{cases}$$

Hence

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \begin{cases} 0 & \text{if } r = 0, \\ 1 & \text{if } r = 1, \\ 0 & \text{if } r > 1. \end{cases}$$

Thus in all cases the formula $f'(x) = rx^{r-1}$ holds. |||

11.2 Corollary (Of the proof of theorem 11.1) *For all $r \in \mathbf{Q}$,*

$$\lim_{x \to 1} \frac{x^r - 1}{x - 1} = r.$$

11.3 Theorem (Derivatives of sin **and** cos.**)** *Let $r \in \mathbf{R}$ and let $f(x) = \sin(rx)$, $g(x) = \cos(rx)$ for all $x \in \mathbf{R}$. Then f and g are differentiable on $\mathbf{R}$, and for all $x \in \mathbf{R}$*

$$\begin{aligned} f'(x) &= r\cos(rx), & (11.4) \\ g'(x) &= -r\sin(rx). & (11.5) \end{aligned}$$

Proof: If $r = 0$ the result is clear, so we assume $r \neq 0$. For all $x \in \mathbf{R}$ and all $t \in \mathbf{R} \setminus \{x\}$, we have

$$\begin{aligned} \frac{\sin(rt) - \sin(rx)}{t - x} &= \frac{2\cos\left(\frac{r(t+x)}{2}\right)\sin\left(\frac{r(t-x)}{2}\right)}{t - x} \\ &= r\cos\left(\frac{r(t+x)}{2}\right) \cdot \frac{\sin\left(\frac{r(t-x)}{2}\right)}{\left(\frac{r(t-x)}{2}\right)}. \end{aligned}$$

(Here I've used an identity from theorem 9.21.) Let $\{x_n\}$ be a generic sequence in $\mathbf{R} \setminus \{x\}$ such that $\{x_n\} \to x$. Let $y_n = \dfrac{r(x_n + x)}{2}$ and let $z_n = \dfrac{r(x_n - x)}{2}$. Then $\{y_n\} \to rx$ so by lemma 9.34 we have $\{\cos(y_n)\} \to \cos(rx)$. Also $\{z_n\} \to 0$, and $z_n \in \mathbf{R} \setminus \{0\}$ for all $n \in \mathbf{Z}^+$, so by (9.38), $\left\{\dfrac{\sin(z_n)}{z_n}\right\} \to 1$. Hence

$$\left\{\frac{\sin(rx_n) - \sin(rx)}{x_n - x}\right\} = \left\{r\cos(y_n) \cdot \frac{\sin(z_n)}{z_n}\right\} \to r\cos(rx),$$

and this proves formula (11.4). ‖

The proof of (11.5) is similar.

11.6 Exercise. Prove that if $g(x) = \cos(rx)$, then $g'(x) = -r\sin(rx)$.

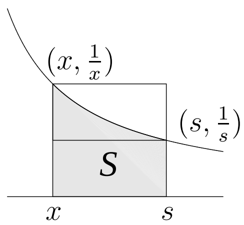
11.7 Theorem (Derivative of the logarithm.) *The logarithm function is differentiable on* $\mathbf{R}^+$*, and*

$$\ln'(x) = \frac{1}{x} \text{ for all } x \in \mathbf{R}^+.$$

Proof: Let $x \in \mathbf{R}^+$, and let $s \in \mathbf{R}^+ \setminus \{x\}$. Then

$$\frac{\ln(s) - \ln(x)}{s - x} = \frac{1}{s-x}\int_x^s \frac{1}{t}dt = \frac{1}{s-x}A_x^s\Big[\frac{1}{t}\Big].$$

Case 1: If $s > x$ then $A_x^s\Big[\frac{1}{t}\Big]$ represents the area of the shaded region S in the figure.



We have

$$B(x, s\colon 0, \frac{1}{s}) \subset S \subset B(x, s\colon 0, \frac{1}{x})$$

so by monotonicity of area

$$\frac{s-x}{s} \le A_x^s\Big[\frac{1}{t}\Big] \le \frac{s-x}{x}.$$

Thus

$$\frac{1}{s} \le \frac{1}{s-x}\int_x^s \frac{1}{t}dt \le \frac{1}{x}. \tag{11.8}$$

Case 2. If $s < x$ we can reverse the roles of s and x in equation (11.8) to get

$$\frac{1}{x} \le \frac{1}{x-s}\int_s^x \frac{1}{t}dt \le \frac{1}{s}$$

or

$$\frac{1}{x} \leq \frac{1}{s-x}\int_x^s \frac{1}{t}dt \leq \frac{1}{s}.$$

In both cases it follows that

$$0 \leq \Big|\frac{1}{s-x}\int_x^s \frac{1}{t}dt - \frac{1}{x}\Big| \leq \Big|\frac{1}{s} - \frac{1}{x}\Big|.$$

Let $\{x_n\}$ be a generic sequence in $\mathbf{R}^+ \setminus \{x\}$ such that $\{x_n\} \to x$. Then $\Big\{\frac{1}{x_n} - \frac{1}{x}\Big\} \to 0$, so by the squeezing rule

$$\Big\{\frac{1}{x_n - x}\int_x^{x_n} \frac{1}{t}dt - \frac{1}{x}\Big\} \to 0,$$

i.e.

$$\left\{\frac{\ln(x_n) - \ln(x)}{x_n - x} - \frac{1}{x}\right\} \to 0.$$

Hence

$$\Big\{\frac{\ln(x_n) - \ln(x)}{x_n - x}\Big\} \to \frac{1}{x}.$$

We have proved that $\ln'(x) = \frac{1}{x}$. ‖

11.9 Assumption (Localization rule for derivatives.) *Let f, g be two real valued functions. Suppose there is some $\epsilon \in \mathbf{R}^+$ and $a \in \mathbf{R}$ such that*

$$(a-\epsilon, a+\epsilon) \subset \text{domain}(f) \cap \text{domain}(g)$$

and such that

$$f(x) = g(x) \text{ for all } x \in (a-\epsilon, a+\epsilon).$$

If f is differentiable at a, then g is differentiable at a and $g'(a) = f'(a)$.

This is another assumption that is really a theorem, i.e. it can be proved. Intuitively this assumption is very plausible. It says that if two functions agree on an entire interval centered at a, then their graphs have the same tangents at a.

11.10 Theorem (Derivative of absolute value.) *Let $f(x) = |x|$ for all $x \in \mathbf{R}$. Then $f'(x) = \frac{x}{|x|}$ for all $x \in \mathbf{R} \setminus \{0\}$ and $f'(0)$ is not defined.*

Proof: Since

$$f(x) = \begin{cases} x & \text{if } x > 0, \\ -x & \text{if } x < 0, \end{cases}$$

it follows from the localization theorem that

$$f'(x) = \begin{cases} 1 = \frac{x}{|x|} & \text{if } x > 0, \\ -1 = \frac{x}{|x|} & \text{if } x < 0. \end{cases}$$

To see that f is not differentable at 0, we want to show that

$$\lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} \frac{|t|}{t}$$

does not exist. Let $x_n = \dfrac{(-1)^n}{n}$. Then $\{x_n\} \to 0$, but $\dfrac{|x_n|}{x_n} = \dfrac{\left|\frac{(-1)^n}{n}\right|}{\frac{(-1)^n}{n}} = (-1)^n$ and we know that $\lim\{(-1)^n\}$ does not exist. Hence $\lim \dfrac{f(t) - f(0)}{t - 0}$ does not exist, i.e., f is not differentiable at 0.

11.11 Definition ($\frac{d}{dx}$ notation for derivatives.) An alternate notation for representing derivatives is:

$$\frac{d}{dx} f(x) = f'(x)$$

or

$$\frac{df}{dx} = f'(x).$$

This notation is used in the following way

$$\begin{aligned} \frac{d}{dx}\Big(\sin(6x)\Big) &= 6\cos(6x), \\ \frac{d}{dt}\Big(\cos(\frac{t}{3})\Big) &= -\frac{1}{3}\sin(\frac{t}{3}). \end{aligned}$$

Or:

Let $f = x^{1/2}$. Then $\dfrac{df}{dx} = \dfrac{1}{2}x^{-1/2}$.

Let $g(x) = \dfrac{1}{x}$. Then $\dfrac{dg}{dx} = \dfrac{d}{dx}\Big(g(x)\Big) = \dfrac{d}{dx}\Big(\dfrac{1}{x}\Big) = -\dfrac{1}{x^2}$.

The $\dfrac{d}{dx}$ notation is due to Leibnitz, and is older than our concept of function.

Leibnitz wrote the differentiation formulas as "$dx^a = ax^{a-1}dx$," or if $y = x^a$, then "$dy = ax^{a-1}dx$." The notation $f'(x)$ for derivatives is due to Joseph Louis Lagrange (1736-1813). Lagrange called $f'(x)$ the *derived function* of $f(x)$ and it is from this that we get our word *derivative*. Leibnitz called derivatives, *differentials* and Newton called them *fluxions*.

Many of the early users of the calculus thought of the derivative as the quotient of two numbers

$$\frac{df}{dx} = \frac{\text{difference in } f}{\text{difference in } x} = \frac{f(x) - f(t)}{x - t}$$

when $dx = x - t$ was "infinitely small". Today "infinitely small" real numbers are out of fashion, but some attempts are being made to bring them back. Cf *Surreal Numbers : How two ex-students turned on to pure mathematics and found total happiness : a mathematical novelette*, by D. E. Knuth.[30]. or *The Hyperreal Line* by H. Jerome Keisler[28, pp 207-237].

11.2 Some General Differentiation Theorems.

11.12 Theorem (Sum rule for derivatives.) *Let f, g be real valued functions with domain(f) $\subset \mathbf{R}$ and domain(g) $\subset \mathbf{R}$, and let $c \in \mathbf{R}$. Suppose f and g are differentiable at a. Then $f + g$, $f - g$ and cf are differentiable at a, and*

$$\begin{aligned}(f+g)'(a) &= f'(a) + g'(a)\\ (f-g)'(a) &= f'(a) - g'(a)\\ (cf)'(a) &= c \cdot f'(a).\end{aligned}$$

Proof: We will prove only the first statement. The proofs of the other statements are similar. For all $x \in \text{dom}(f)$ we have

$$\begin{aligned}\frac{(f+g)(x) - (f+g)(a)}{x-a} &= \frac{f(x) + g(x) - (f(a) + g(a))}{x-a}\\ &= \frac{f(x) - f(a)}{x-a} + \frac{g(x) - g(a)}{x-a}\end{aligned}$$

By the sum rule for limits of functions, it follows that

$$\lim_{x\to a}\left(\frac{(f+g)(x) - (f+g)(a)}{x-a}\right) = \lim_{x\to a}\left(\frac{f(x) - f(a)}{x-a}\right) + \lim_{x\to a}\left(\frac{g(x) - g(a)}{x-a}\right),$$

i.e.

$$(f+g)'(a) = f'(a) + g'(a). \parallel$$

11.13 Examples. If

$$f(x) = 27x^3 + \frac{1}{3x} + \sqrt{8x},$$

then

$$f(x) = 27x^3 + \frac{1}{3}x^{-1} + \sqrt{8} \cdot x^{1/2},$$

so

$$\begin{aligned} f'(x) &= 27 \cdot (3x^2) + \frac{1}{3}(-1 \cdot x^{-2}) + \sqrt{8} \cdot (\frac{1}{2}x^{-1/2}) \\ &= 81x^2 - \frac{1}{3x^2} + \sqrt{\frac{2}{x}}. \end{aligned}$$

If $g(x) = (3x^2+7)^2$, then $g(x) = 9x^4 + 42x^2 + 49$, so

$$g'(x) = 9 \cdot 4x^3 + 42 \cdot 2x = 36x^3 + 84x.$$

If $h(x) = \sin(4x) + \sin^2(4x)$, then $h(x) = \sin(4x) + \frac{1}{2}\Big(1 - \cos(8x)\Big)$, so

$$\begin{aligned} h'(x) &= 4\cos(4x) + \frac{1}{2}(-1)\Big(-8 \cdot \sin(8x)\Big) \\ &= 4\cos(4x) + 4\sin(8x). \end{aligned}$$

$$\frac{d}{ds}\Big(8\sin(4s) + s^2 + 4\Big) = 32\cos(4s) + 2s.$$

11.14 Exercise. Calculate the derivatives of the following functions:

a) $f(x) = (x^2+4x)^2$

b) $g(x) = \sqrt{3x^3} + \dfrac{4}{x^4}$

c) $h(t) = \ln(t) + \ln(t^2) + \ln(t^3)$

d) $k(x) = \ln(10 \cdot x^{5/2})$

e) $l(x) = 3\cos(x) + \cos(3x)$

f) $m(x) = \cos(x)\cos(3x)$

g) $n(x) = \Big(\sin^2(x) + \cos^2(x)\Big)^4$

11.15 Exercise. Calculate

a) $\dfrac{d}{dt}\Big(1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \dfrac{t^4}{4!}\Big)$

b) $\dfrac{d}{dt}\Big(h_0 + v_0(t - t_0) - \dfrac{1}{2}g(t - t_0)^2\Big)$. Here h_0, v_0, t_0 and g are all constants.

c) $\dfrac{d}{dt}(|-100t|)$

11.16 Theorem (The product rule for derivatives.) *Let f and g be real valued functions with* $\mathrm{dom}(f) \subset \mathbf{R}$ *and* $\mathrm{dom}(g) \subset \mathbf{R}$*. Suppose f and g are both differentiable at a. Then fg is differentiable at a and*

$$(fg)'(a) = f(a) \cdot g'(a) + f'(a) \cdot g(a).$$

In particular, if $f = c$ is a constant function, we have

$$(cf)'(a) = c \cdot f'(a).$$

Proof: Let x be a generic point of $\mathrm{dom}(f) \cap \mathrm{dom}(g) \setminus \{a\}$. Then

$$\begin{aligned}\frac{(fg)(x) - (fg)(a)}{x - a} &= \frac{f(x)\big(g(x) - g(a)\big) + \big(f(x) - f(a)\big)g(a)}{x - a} \\ &= f(x)\left(\frac{g(x) - g(a)}{x - a}\right) + \left(\frac{f(x) - f(a)}{x - a}\right)g(a).\end{aligned}$$

We know that $\lim\limits_{x \to a}\Big(\dfrac{g(x) - g(a)}{x - a}\Big) = g'(a)$ and $\lim\limits_{x \to a}\Big(\dfrac{f(x) - f(a)}{x - a}\Big) = f'(a)$. If we also knew that $\lim\limits_{x \to a} f(x) = f(a)$, then by basic properties of limits we could say that

$$(fg)'(a) = \lim_{x \to a}\frac{(fg)(x) - (fg)(a)}{x - a} = f(a)g'(a) + f'(a)g(a)$$

which is what we claimed.

This missing result will be needed in some other theorems, so I've isolated it in the following lemma.

11.17 Lemma (Differentiable functions are continuous.) *Let f be a real valued function such that* $\text{dom}(f) \subset \mathbf{R}^+$*. Suppose f is differentiable at a point $a \in$* $\text{dom}(f)$*. Then $\lim_{x \to a} f(x) = f(a)$. (We will define "continuous" later. Note that neither the statement nor the proof of this lemma use the word "continuous" in spite of the name of the lemma.)*

Proof:

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) \right).$$

Hence by the product and sum rules for limits,

$$\lim_{x \to a} f(x) = f'(a) \cdot (a - a) + f(a) = f(a). \ \|$$

11.18 Example (Leibniz's proof of the product rule.) Leibniz stated the product rule as

$$dxy = xdy + ydx[34, page143]^1$$

His proof is as follows:

> dxy is the difference between two successive xy's; let one of these be xy and the other $x + dx$ into $y + dy$; then we have
>
> $$dxy = \overline{\overline{\overline{x + dx} \cdot \overline{y + dy}}} - xy = xdy + ydx + dxdy;$$
>
> the omission of the quantity $dxdy$ which is infinitely small in comparison with the rest, for it is supposed that dx and dy are infinitely small (because the lines are understood to be continuously increasing or decreasing by very small increments throughout the series of terms), will leave $xdy + ydx$.[34, page 143]

Notice that for Leibniz, the important thing is not the *derivative*, $\dfrac{dxy}{dt}$, but the infinitely small *differential*, dxy.

[1] The actual statement is $dxy = xdx + ydy$, but this is a typographical error, since the proof gives the correct formula.

11.19 Theorem (Derivative of a reciprocal.) *Let f be a real valued function such that* $\text{dom}(f) \subset \mathbf{R}$*. Suppose f is differentiable at some point a, and $f(a) \neq 0$. Then $\frac{1}{f}$ is differentiable at a, and*

$$\Big(\frac{1}{f}\Big)'(a) = \frac{-f'(a)}{\big(f(a)\big)^2}.$$

Proof: For all $x \in \text{dom}\Big(\frac{1}{f}\Big) \setminus \{a\}$

$$\frac{\frac{1}{f(x)} - \frac{1}{f(a)}}{x-a} = \frac{f(a)-f(x)}{(x-a)f(x)f(a)} = -\frac{\big(f(x)-f(a)\big)}{(x-a)} \cdot \frac{1}{f(x)f(a)}.$$

It follows from the standard limit rules that

$$\lim_{x \to a} \frac{\frac{1}{f(x)} - \frac{1}{f(a)}}{x-a} = -f'(a) \cdot \frac{1}{\big(f(a)\big)^2}.$$

11.20 Theorem (Quotient rule for derivatives.) *Let f, g be real valued functions with* $\text{dom}(f) \subset \mathbf{R}$ *and* $\text{dom}(g) \subset \mathbf{R}$*. Suppose f and g are both differentiable at a, and that $g(a) \neq 0$. Then $\frac{f}{g}$ is differentiable at a, and*

$$\Big(\frac{f}{g}\Big)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{\big(g(a)\big)^2}.$$

11.21 Exercise. Prove the quotient rule.

11.22 Examples. Let

$$f(x) = \frac{\sin(x)}{x} \text{ for } x \in \mathbf{R} \setminus \{0\}.$$

Then by the quotient rule

$$f'(x) = \frac{x(\cos(x)) - \sin(x)}{x^2}.$$

Let $h(x) = x^2 \cdot |x|$. Then by the product rule

$$h'(x) = x^2\Big(\frac{x}{|x|}\Big) + 2x|x| = x|x| + 2x|x| = 3x|x|$$

(since $\dfrac{x^2}{|x|} = \dfrac{|x|^2}{|x|} = |x|$).

The calculation is not valid at $x = 0$ (since $|x|$ is not differentiable at 0, and we divided by $|x|$ in the calculation. However h is differentiable at 0 since $\displaystyle\lim_{t\to 0}\frac{h(t)-h(0)}{t-0} = \lim_{t\to 0}\frac{t^2|t|}{t} = \lim_{t\to 0} t|t| = 0$, i.e., $h'(0) = 0 = 3\cdot 0\cdot |0|$. Hence the formula

$$\frac{d}{dx}(x^2|x|) = 3x|x|$$

is valid for all $x \in \mathbf{R}$.

Let $g(x) = \ln(x)\cdot\sin(10x)\cdot\sqrt{x}$. Consider g to be a product $g = hk$ where $h(x) = \ln(x)\cdot\sin(10x)$ and $k(x) = \sqrt{x}$. Then we can apply the product rule twice to get

$$\begin{aligned} g'(x) &= \Big(\ln(x)\cdot\sin(10x)\Big)\cdot\frac{1}{2\sqrt{x}} \\ &\quad + \Bigg(\ln(x)\cdot\Big(10\cos(10x)\Big) + \frac{1}{x}\sin(10x)\Bigg)\sqrt{x}. \end{aligned}$$

11.23 Exercise (Derivatives of tangent, cotangent, secant, cosecant.) We define functions tan, cot, sec, and csc by

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)},$$

$$\sec(x) = \frac{1}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}.$$

The domains of these functions are determined by the definition of the domain of a quotient, e.g. $\text{dom}(\sec) = \{x \in \mathbf{R} \colon \cos x \neq 0\}$. Prove that

$$\begin{aligned} &\tfrac{d}{dx}\tan(x) = \sec^2(x), && \tfrac{d}{dx}\cot(x) = -\csc^2 x, \\ &\tfrac{d}{dx}\sec(x) = \tan(x)\sec(x), && \tfrac{d}{dx}\csc(x) = -\cot(x)\csc(x). \end{aligned}$$

(You should memorize these formulas. Although they are easy to derive, later we will want to use them backwards; i.e., we will want to find a function whose derivative is $\sec^2(x)$. It is not easy to derive the formulas backwards.)

11.24 Exercise. Calculate the derivatives of the following functions. Simplify your answers if you can.

a) $f(x) = x \cdot \ln(x) - x$.

b) $g(x) = \dfrac{ax+b}{cx+d}$ (here a, b, c, d are constants).

c) $k(x) = (x^2 + 3x + 10)(x^2 + 3x + 12)$.

d) $m(x) = \dfrac{\cos(6x)}{\cos(7x)}$.

11.25 Exercise. Let f, g, h, and k be differentiable functions defined on $\mathbf{R}$.
a) Express $(fgh)'$ in terms of f, f', g, g', h and h'.
b) On the basis of your answer for part a), try to guess a formula for $(fghk)'$. Then calculate $(fghk)'$, and see whether your guess was right.

11.3 Composition of Functions

11.26 Definition ($f \circ g$.) Let A, B, C, D be sets and let $f\colon A \to B,\ g\colon C \to D$ be functions. The composition of f and g is the function $f \circ g$ defined by:

$$\begin{aligned} \operatorname{codomain}(f \circ g) &= B = \operatorname{codomain}(f). \\ \operatorname{dom}(f \circ g) &= \{x \in C\colon g(x) \in A\} \\ &= \{x \in \operatorname{dom}(g)\colon g(x) \in \operatorname{dom}(f)\}; \end{aligned}$$

i.e., $\operatorname{dom}(f \circ g)$ is the set of all points x such that $f\big(g(x)\big)$ is defined. The rule for $f \circ g$ is

$$(f \circ g)(x) = f\big(g(x)\big) \text{ for all } x \in \operatorname{dom}(f \circ g).$$

11.27 Example. If $f(x) = \sin(x)$ and $g(x) = x^2 - 2$, then

$$(f \circ g)(x) = \sin(x^2 - 2)$$

and

$$(g \circ f)(x) = \sin^2(x) - 2.$$

Thus

$$(f \circ g)(0) = \sin(-2) \text{ and } (g \circ f)(0) = -2 \neq (f \circ g)(0).$$

So in this case $f \circ g \neq g \circ f$. Thus composition is not a commutative operation.

If $h(x) = \ln(x)$ and $k(x) = |x|$, then

$$(h \circ k)(x) = \ln(|x|)$$

and

$$(k \circ h)(x) = |\ln(x)|.$$

11.28 Exercise. For each of the functions F below, find functions f and g such that $F = f \circ g$. Then find a formula for $g \circ f$.

a) $F(x) = \ln(\tan(x))$.

b) $F(x) = \sin(4(x^2 + 3))$.

c) $F(x) = |\sin(x)|$.

11.29 Exercise. Let

$$\begin{aligned} f(x) &= \sqrt{1 - x^2}, \\ g(x) &= \frac{1}{1 - x}. \end{aligned}$$

Calculate formulas for $f \circ f$, $f \circ (f \circ f)$, $(f \circ f) \circ f$, $g \circ g$, $(g \circ g) \circ g$, and $g \circ (g \circ g)$.

11.30 Entertainment (Composition problem.) From the previous exercise you should be able to find a subset A of $\mathbf{R}$, and a function $f : A \to \mathbf{R}$ such that $(f \circ f)(x) = x$ for all $x \in A$. You should also be able to find a subset B of $\mathbf{R}$ and a function $g : B \to \mathbf{R}$ such that $(g \circ (g \circ g))(x) = x$ for all $x \in B$. Can you find a subset C of $\mathbf{R}$, and a function $h : C \to \mathbf{R}$ such that $(h \circ (h \circ (h \circ h)))(x) = x$ for all $x \in C$? One obvious example is the function f from the previous example. To make the problem more interesting, also add the condition that $(h \circ h)(x) \neq x$ for some x in C.

11.31 Theorem (Chain rule.) *Let f, g be real valued functions such that* $\mathrm{dom}(f) \subset \mathbf{R}$ *and* $\mathrm{dom}(g) \subset \mathbf{R}$. *Suppose $a \in \mathrm{dom}(g)$ and $g(a) \in \mathrm{dom} f$, and g is differentiable at a and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable at a, and*

$$(f \circ g)'(a) = f'\big(g(a)\big) \cdot g'(a).$$

Before we prove the theorem we will give a few examples of how it is used:

11.32 Example. Let $H(x) = \sqrt{10 + \sin x}$. Then $H = f \circ g$ where

$$f(x) = \sqrt{x}, \qquad g(x) = 10 + \sin(x),$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad g'(x) = \cos(x).$$

Hence

$$\begin{aligned} H'(x) &= f'\big(g(x)\big) \cdot g'(x) \\ &= \frac{1}{2\sqrt{10 + \sin(x)}} \cdot \cos(x). \end{aligned}$$

Let $K(x) = \ln(5x^2 + 1)$. Then $K = f \circ g$ where

$$f(x) = \ln(x), \quad g(x) = 5x^2 + 1,$$

$$f'(x) = \frac{1}{x}, \qquad g'(x) = 10x.$$

Hence

$$\begin{aligned} K'(x) &= f'\big(g(x)\big) \cdot g'(x) \\ &= \frac{1}{5x^2 + 1} \cdot 10x = \frac{10x}{5x^2 + 1}. \end{aligned}$$

Usually I will not write out all of the details of a calculation like this. I will just write:

Let $f(x) = \tan(2x + 4)$. Then $f'(x) = \sec^2(2x + 4) \cdot 2$.

Proof of chain rule: Suppose g is differentiable at a and f is differentiable at $g(a)$. Then

$$\frac{f\Big(g(x)\Big) - f\Big(g(a)\Big)}{x-a} = \frac{f\Big(g(x)\Big) - f\Big(g(a)\Big)}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a}. \tag{11.33}$$

Since g is differentiable at a, we know that

$$\lim_{x\to a} \frac{g(x)-g(a)}{x-a} = g'(a).$$

Hence the theorem will follow from (11.33), the definition of derivative, and the product rule for limits of functions, if we can show that

$$\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} = f'(g(a)).$$

Since g is differentiable at a, it follows from lemma 11.17 that

$$\lim_{x\to a} g(x) = g(a). \tag{11.34}$$

Let $\{x_n\}$ be a generic sequence in $\text{dom}(f\circ g)\setminus\{a\}$, such that $\{x_n\}\to a$. Then by (11.34), we have

$$\lim\{g(x_n)\} = g(a). \tag{11.35}$$

Since f is differentiable at $g(a)$, we have

$$\lim_{t\to g(a)} \frac{f(t) - f(g(a))}{t-g(a)} = f'(g(a)).$$

From this and (11.35) it follows that

$$\lim\left\{\frac{f(g(x_n)) - f(g(a))}{g(x_n)-g(a)}\right\} = f'(g(a)).$$

Since this holds for a generic sequence $\{x_n\}$ in $\text{dom}(f\circ g)\setminus\{a\}$, we have

$$\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a} = f'(g(a)),$$

which is what we wanted to prove. To complete the proof, I should show that a is an interior point of $\text{dom}(f\circ g)$. This turns out to be rather tricky, so I will omit the proof.

Remark: Our proof of the chain rule is not valid in all cases, but it is valid in all cases where you are likely to use it. The proof fails in the case where every interval $\Big(g(a)-\epsilon,\ g(a)+\epsilon\Big)$ contains a point $b \neq a$ for which $g(b) = g(a)$. (You should check the proof to see where this assumption was made.) Constant functions g satisfy this condition, but if g is constant then $f \circ g$ is also constant so the chain rule holds trivially in this case. Since the proof in the general case is more technical than illuminating, I am going to omit it. Can you find a non-constant function g for which the proof fails?

11.36 Example. If f is differentiable at x, and $f(x) \neq 0$, then

$$\frac{d}{dx}\Big(|f(x)|\Big) = \frac{f(x)}{|f(x)|} f'(x).$$

Also

$$\begin{aligned}\frac{d}{dx}\left(\ln\Big(|f(x)|\Big)\right) &= \frac{1}{|f(x)|}\frac{d}{dx}\Big(|f(x)|\Big)\\ &= \frac{1}{|f(x)|}\frac{f(x)}{|f(x)|}f'(x) = \frac{f(x)f'(x)}{f(x)^2} = \frac{f'(x)}{f(x)};\end{aligned}$$

i.e.,

$$\frac{d}{dx}\Big(\ln|f(x)|\Big) = \frac{f'(x)}{f(x)} \tag{11.37}$$

I will use this relation frequently.

11.38 Example (Logarithmic differentiation.) Let

$$h(x) = \frac{\sqrt{(x^2+1)}(x^2-4)^{10}}{(x^3+x+1)^3}. \tag{11.39}$$

The derivative of h can be found by using the quotient rule and the product rule and the chain rule. I will use a trick here which is frequently useful. I have

$$\ln\Big(|h(x)|\Big) = \frac{1}{2}\ln(x^2+1) + 10\ln(|x^2-4|) - 3\ln(|x^3+x+1|).$$

Now differentiate both sides of this equation using (11.37) to get

$$\frac{h'(x)}{h(x)} = \frac{1}{2}\frac{2x}{x^2+1} + 10\frac{2x}{x^2-4} - 3\frac{3x^2+1}{x^3+x+1}.$$

Multiply both sides of the equation by $h(x)$ to get

$$h'(x) = \frac{\sqrt{x^2+1}(x^2-4)^{10}}{(x^3+x+1)^3}\Big[\frac{x}{x^2+1} + \frac{20x}{x^2-4} - \frac{3(3x^2+1)}{x^3+x+1}\Big].$$

This formula is not valid at points where $h(x) = 0$, because we took logarithms in the calculation. Thus h is differentiable at $x = 2$, but our formula for $h'(x)$ is not defined when $x = 2$.

The process of calculating f' by first taking the logarithm of the absolute value of f and then differentiating the result, is called *logarithmic differentiation.*

11.40 Exercise. Let h be the function defined in (11.39) Show that h is differentiable at 2, and calculate $h'(2)$.

11.41 Exercise. Find derivatives for the functions below. (Assume here that f is a function that is differentiable at all points being considered.)

a) $F(x) = \sin(f(x))$.

b) $G(x) = \cos(f(x))$.

c) $H(x) = (f(x))^r$, where r is a rational number.

d) $K(x) = \ln((f(x))$.

e) $L(x) = |f(x)|$.

f) $M(x) = \tan(f(x))$.

g) $N(x) = \cot(f(x))$.

h) $P(x) = \sec(f(x))$.

i) $Q(x) = \csc(f(x))$.

j) $R(x) = \ln(|f(x)|)$.

11.42 Exercise. Find derivatives for the functions below. (Assume here that f is a function that is differentiable at all points being considered.)

a) $F(x) = f(\sin(x))$.

b) $G(x) = f(\cos(x))$.

c) $H(x) = f(x^r)$, where r is a rational number.

d) $K(x) = f(\ln(x))$.

e) $L(x) = f(|x|)$.

f) $M(x) = f(\tan(x))$.

g) $N(x) = f(\cot(x))$.

h) $P(x) = f(\sec(x))$.

i) $Q(x) = f(\csc(x))$.

j) $R(x) = f(\ln(|x|)$.

11.43 Exercise. Calculate the derivatives of the following functions. Simplify your answers.

a) $a(x) = \sin^3(x) = (\sin(x))^3$.

b) $b(x) = \sin(x^3)$.

c) $c(x) = (x^2 + 4)^{10}$.

d) $f(x) = \sin(4x^2 + 3x)$.

e) $g(x) = \ln\Big(|\cos(x)|\Big)$.

f) $h(x) = \ln\Big(|\sec(x)|\Big)$.

g) $k(x) = \ln\Big(|\sec(x) + \tan(x)|\Big)$.

h) $l(x) = \ln\Big(|\csc(x) + \cot(x)|\Big)$.

i) $m(x) = 3x^3\ln(5x) - x^3$.

j) $n(x) = \sqrt{x^2+1} + \ln\Big(\dfrac{\sqrt{x^2+1} - 1}{x}\Big)$.

k) $p(x) = \dfrac{1}{2}(x+4)^2 - 8x + 16\ln(x+4)$.

l) $q(x) = \dfrac{x}{2}\Big[\sin\Big(\ln(|6x|)\Big) - \cos\Big(\ln(|6x|)\Big)\Big].$