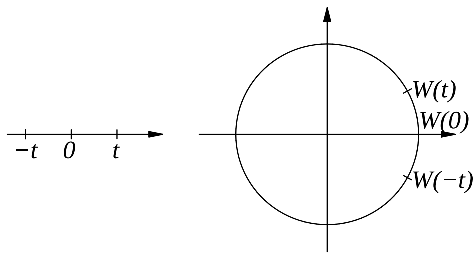
Chapter 9

Trigonometric Functions

9.1 Properties of Sine and Cosine

9.1 Definition ($W(t)$.) We define a function $W: \mathbf{R} \to \mathbf{R}^2$ as follows.

If $t \geq 0$, then $W(t)$ is the point on the unit circle such that the length of the arc joining $(1,0)$ to $W(t)$ (measured in the counterclockwise direction) is equal to t. (There is an optical illusion in the figure. The length of segment $[0,t]$ is equal to the length of arc $W(0)W(t)$.)



Thus to find $W(t)$, you should start at $(1,0)$ and move along the circle in a counterclockwise direction until you've traveled a distance t. Since the circumference of the circle is 2π, we see that $W(2\pi) = W(4\pi) = W(0) = (1,0)$. (Here we assume Archimedes' result that the area of a circle is half the circumference times the radius.) If $t < 0$, we define

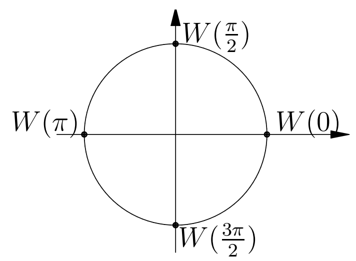
$$W(t) = H(W(-t)) \text{ for } t < 0 \tag{9.2}$$

where H is the reflection about the horizontal axis. Thus if $t < 0$, then $W(t)$ is the point obtained by starting at $(1,0)$ and moving $|t|$ along the unit circle in the clockwise direction.

Remark: The definition of W depends on several ideas that we have not defined or stated assumptions about, e.g., *length of an arc* and *counterclockwise direction.* I believe that the amount of work it takes to formalize these ideas at this point is not worth the effort, so I hope your geometrical intuition will carry you through this chapter. (In this chapter we will assume quite a bit of Euclidean geometry, and a few properties of area that do not follow from our assumptions stated in chapter 5.)

A more self contained treatment of the trigonometric functions can be found in [44, chapter 15], but the treatment given there uses ideas that we will consider later, (e.g. derivatives, inverse functions, the intermediate value theorem, and the fundamental theorem of calculus) in order to *define* the trigonometric functions.

We have the following values for W:



$$W(0) = (1,0) \tag{9.3}$$
$$W\left(\frac{\pi}{2}\right) = (0,1) \tag{9.4}$$
$$W(\pi) = (-1,0) \tag{9.5}$$
$$W\left(\frac{3\pi}{2}\right) = (0,-1) \tag{9.6}$$
$$W(2\pi) = (1,0) = W(0). \tag{9.7}$$

In general

$$W(t + 2\pi k) = W(t) \text{ for all } t \in \mathbf{R} \text{ and all } k \in \mathbf{Z}. \tag{9.8}$$

9.9 Definition (Sine and cosine.) In terms of coordinates, we write

$$W(t) = \Big(\cos(t), \sin(t)\Big).$$

(We read "cos(t)" as "cosine of t", and we read "sin(t)" as "sine of t".)

Since $W(t)$ is on the unit circle, we have

$$\sin^2(t) + \cos^2(t) = 1 \text{ for all } t \in \mathbf{R},$$

and

$$-1 \le \sin t \le 1, \quad -1 \le \cos t \le 1 \text{ for all } t \in \mathbf{R}.$$

The equations (9.3) - (9.8) show that

$$\begin{aligned}
&\cos(0) = 1, && \sin(0) = 0,\\
&\cos\left(\tfrac{\pi}{2}\right) = 0, && \sin\left(\tfrac{\pi}{2}\right) = 1,\\
&\cos(\pi) = -1, && \sin(\pi) = 0,\\
&\cos\left(\tfrac{3\pi}{2}\right) = 0, && \sin\left(\tfrac{3\pi}{2}\right) = -1,
\end{aligned}$$

and

$$\begin{aligned}
\cos(t + 2\pi k) &= \cos t \quad \text{for all } t \in \mathbf{R} \text{ and all } k \in \mathbf{Z},\\
\sin(t + 2\pi k) &= \sin t \quad \text{for all } t \in \mathbf{R} \text{ and all } k \in \mathbf{Z}.
\end{aligned}$$

In equation (9.2) we defined

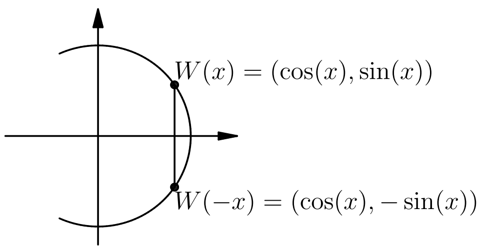
$$W(t) = H(W(-t)) \text{ for } t < 0.$$

Thus for $t < 0$,

$$W(-t) = H(H(W(-t))) = H(W(t)) = H(W(-(-t))),$$

and it follows that

$$W(t) = H(W(-t)) \text{ for all } t \in \mathbf{R}.$$



In terms of components

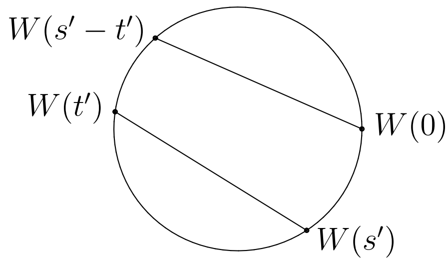
$$\Big(\cos(-t), \sin(-t)\Big) = W(-t) = H(W(t)) = H(\cos(t), \sin(t)) = \Big(\cos(t), -\sin(t)\Big)$$

and consequently

$$\cos(-t) = \cos(t) \text{ and } \sin(-t) = -\sin(t) \text{ for all } t \in \mathbf{R}.$$

Let s, t be arbitrary real numbers. Then there exist integers k and l such that $s + 2\pi k \in [0, 2\pi)$ and $t + 2\pi l \in [0, 2\pi)$. Let

$$s' = s + 2\pi k \text{ and } t' = t + 2\pi l.$$



Then $s' - t' = (s - t) + 2\pi(k - l)$, so

$$W(s) = W(s'),\ W(t) = W(t'),\ W(s - t) = W(s' - t').$$

Suppose $t' \le s'$ (see figure). Then the length of the arc joining $W(s')$ to $W(t')$ is $s' - t'$ which is the same as the length of the arc joining $(1, 0)$ to $W(s' - t')$. Since equal arcs in a circle subtend equal chords, we have

$$\text{dist}\Big(W(s'), W(t')\Big) = \text{dist}\Big(W(s' - t'), (1, 0)\Big)$$

and hence

$$\text{dist}\Big(W(s), W(t)\Big) = \text{dist}\Big(W(s - t), (1, 0)\Big). \tag{9.10}$$

You can verify that this same relation holds when $s' < t'$.

9.11 Theorem (Addition laws for sine and cosine.) *For all real numbers s and t,*

$$\cos(s + t) = \cos(s)\cos(t) - \sin(s)\sin(t) \tag{9.12}$$

$$\cos(s - t) = \cos(s)\cos(t) + \sin(s)\sin(t) \tag{9.13}$$

$$\sin(s + t) = \sin(s)\cos(t) + \cos(s)\sin(t) \tag{9.14}$$

$$\sin(s - t) = \sin(s)\cos(t) - \cos(s)\sin(t). \tag{9.15}$$

Proof: From (9.10) we know

$$\mathrm{dist}\Big(W(s), W(t)\Big) = \mathrm{dist}\Big(W(s-t), (1,0)\Big),$$

i.e.,

$$\mathrm{dist}\Big((\cos(s), \sin(s)), (\cos(t), \sin(t))\Big) = \mathrm{dist}\Big((\cos(s-t), \sin(s-t)), (1,0)\Big).$$

Hence

$$\Big(\cos(s) - \cos(t)\Big)^2 + \Big(\sin(s) - \sin(t)\Big)^2 = \Big(\cos(s-t) - 1\Big)^2 + \Big(\sin(s-t)\Big)^2.$$

By expanding the squares and using the fact that $\sin^2(u) + \cos^2(u) = 1$ for all u, we conclude that

$$\cos(s)\cos(t) + \sin(s)\sin(t) = \cos(s-t). \qquad (9.16)$$

This is equation (9.13). To get equation (9.12) replace t by $-t$ in (9.16). If we take $s = \frac{\pi}{2}$ in equation (9.16) we get

$$\cos\Big(\frac{\pi}{2}\Big)\cos(t) + \sin\Big(\frac{\pi}{2}\Big)\sin(t) = \cos\Big(\frac{\pi}{2} - t\Big)$$

or

$$\sin(t) = \cos\Big(\frac{\pi}{2} - t\Big) \text{ for all } t \in \mathbf{R}.$$

If we replace t by $\Big(\frac{\pi}{2} - t\Big)$ in this equation we get

$$\sin\Big(\frac{\pi}{2} - t\Big) = \cos\Big(\frac{\pi}{2} - (\frac{\pi}{2} - t)\Big) = \cos t \text{ for all } t \in \mathbf{R}.$$

Now in equation (9.16) replace s by $\frac{\pi}{2} - s$ and get

$$\cos\Big(\frac{\pi}{2} - s\Big)\cos(t) + \sin\Big(\frac{\pi}{2} - s\Big)\sin(t) = \cos\Big(\frac{\pi}{2} - s - t\Big)$$

or

$$\sin s \cos t + \cos s \sin t = \sin(s+t),$$

which is equation (9.14). Finally replace t by $-t$ in this last equation to get (9.15). |||

In the process of proving the last theorem we proved the following:

9.17 Theorem (Reflection law for sin and cos.) *For all* $x \in \mathbf{R}$*,*

$$\cos(x) = \sin(\frac{\pi}{2} - x) \text{ and } \sin(x) = \cos(\frac{\pi}{2} - x). \tag{9.18}$$

9.19 Theorem (Double angle and half angle formulas.) *For all* $t \in \mathbf{R}$ *we have*

$$\begin{aligned}
&\sin(2t) = 2\sin t\cos t,\\
&\cos(2t) = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 = 1 - 2\sin^2 t,\\
&\sin^2\Big(\frac{t}{2}\Big) = \frac{1-\cos t}{2},\\
&\cos^2\Big(\frac{t}{2}\Big) = \frac{1+\cos t}{2}.
\end{aligned}$$

9.20 Exercise. Prove the four formulas stated in theorem 9.19.

9.21 Theorem (Products and differences of sin and cos.) *For all* s, t *in* $\mathbf{R}$*,*

$$\begin{aligned}
\cos(s)\cos(t) &= \frac{1}{2}[\cos(s-t) + \cos(s+t)], & (9.22)\\
\cos(s)\sin(t) &= \frac{1}{2}[\sin(s+t) - \sin(s-t)], & (9.23)\\
\sin(s)\sin(t) &= \frac{1}{2}[\cos(s-t) - \cos(s+t)], & (9.24)\\
\cos(s) - \cos(t) &= -2\sin\Big(\frac{s+t}{2}\Big)\sin\Big(\frac{s-t}{2}\Big), & (9.25)\\
\sin(s) - \sin(t) &= 2\cos\Big(\frac{s+t}{2}\Big)\sin\Big(\frac{s-t}{2}\Big). & (9.26)
\end{aligned}$$

Proof: We have

$$\cos(s+t) = \cos(s)\cos(t) - \sin(s)\sin(t)$$

and

$$\cos(s-t) = \cos(s)\cos(t) + \sin(s)\sin(t).$$

By adding these equations, we get (9.22). By subtracting the first from the second, we get (9.24).

In equation (9.24) replace s by $\frac{s+t}{2}$ and replace t by $\frac{t-s}{2}$ to get

$$\sin\Big(\frac{s+t}{2}\Big)\sin\Big(\frac{t-s}{2}\Big) = \frac{1}{2}\Big[\cos\Big(\frac{s+t}{2} - \frac{t-s}{2}\Big) - \cos\Big(\frac{s+t}{2} + \frac{t-s}{2}\Big)\Big]$$

or

$$-\sin\Big(\frac{s+t}{2}\Big)\sin\Big(\frac{s-t}{2}\Big) = \frac{1}{2}[\cos(s) - \cos(t)].$$

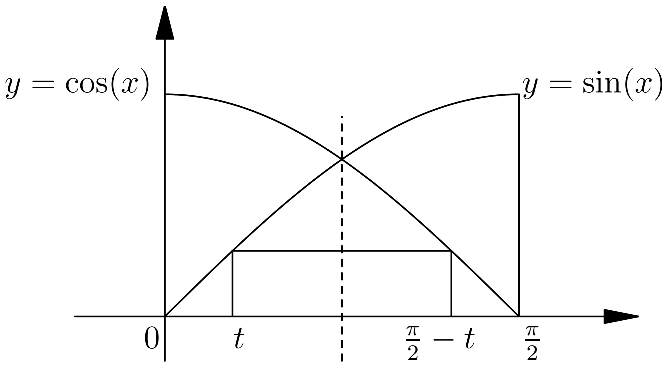
This yields equation (9.25).

9.27 Exercise. Prove equations (9.23) and (9.26).

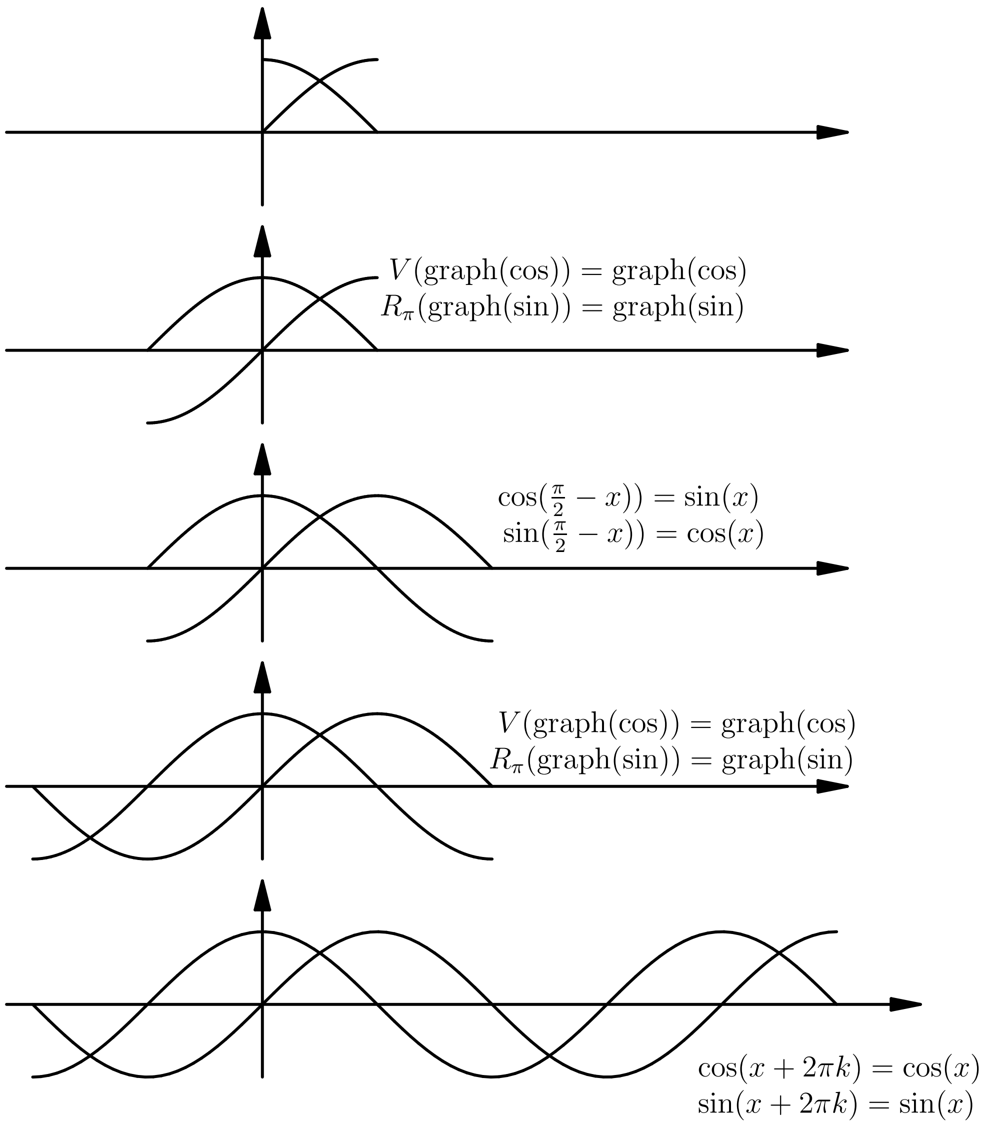
From the geometrical description of sine and cosine, it follows that as t increases for 0 to $\frac{\pi}{2}$, $\sin(t)$ increases from 0 to 1 and $\cos(t)$ decreases from 1 to 0. The identities

$$\sin\Big(\frac{\pi}{2} - t\Big) = \cos(t) \text{ and } \cos\Big(\frac{\pi}{2} - t\Big) = \sin(t)$$

indicate that a reflection about the vertical line through $x = \frac{\pi}{4}$ carries the graph of sin onto the graph of cos, and vice versa.



$\cos(\frac{\pi}{2} - t) = \sin(t)$ $\sin(\frac{\pi}{2} - t) = \cos(t)$



The condition $\cos(-x) = \cos x$ indicates that the reflection about the vertical axis carries the graph of cos to itself.

The relation $\sin(-x) = -\sin(x)$ shows that

$$\begin{aligned}(x, y) \in \text{graph}(\sin) &\implies y = \sin(x)\\ &\implies -y = -\sin(x) = \sin(-x)\\ &\implies (-x, -y) = (-x, \sin(-x))\\ &\implies (-x, -y) \in \text{graph}(\sin)\\ &\implies R_\pi(x, y) \in \text{graph}(\sin)\end{aligned}$$

i.e., the graph of sin is carried onto itself by a rotation through π about the origin.
We have

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$

and $1 = \sin^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right) = 2\cos^2\left(\frac{\pi}{4}\right)$, so $\cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}$ and $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = .707$ (approximately).

With this information we can make a reasonable sketch of the graph of sin and cos (see page 197).

9.28 Exercise. Show that

$$\cos(3x) = 4\cos^3(x) - 3\cos(x) \text{ for all } x \in \mathbf{R}.$$

9.29 Exercise. Complete the following table of sines and cosines:

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
sin	0		$\frac{\sqrt{2}}{2}$		1				0
cos	1		$\frac{\sqrt{2}}{2}$		0				-1

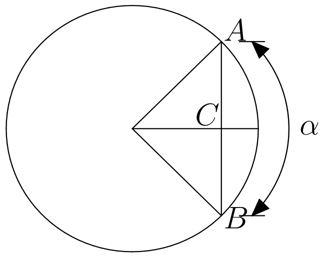
	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin	0				-1				0
cos	-1				0				1

$$\frac{\sqrt{2}}{2} = .707$$

Include an explanation for how you found $\sin\frac{\pi}{6}$ and $\cos\frac{\pi}{6}$ (or $\sin\frac{\pi}{3}$ and $\cos\frac{\pi}{3}$).

For the remaining values you do not need to include an explanation.

Most of the material from this section was discussed by Claudius Ptolemy (fl. 127-151 AD). The functions considered by Ptolemy were not the sine and cosine, but the *chord*, where the chord of an arc α is the length of the segment joining its endpoints.



$AB = \text{chord}(\alpha) \qquad AC = \sin(\frac{\alpha}{2})$

$$\text{chord }(\alpha) = 2\sin(\frac{\alpha}{2}). \tag{9.30}$$

Ptolemy's chords are functions of arcs (measured in degrees), not of numbers. Ptolemy's addition law for sin was (roughly)

$$D\cdot\ \text{chord}(\beta-\alpha) = \text{chord}(\beta)\text{chord}(180^\circ-\alpha) - \text{chord}(180^\circ-\beta)\text{chord}(\alpha),$$

where D is the diameter of the circle, and $0^\circ < \alpha < \beta < 180^\circ$. Ptolemy produced tables equivalent to tables of $\sin(\alpha)$ for $\left(\frac{1}{4}\right)^\circ \leq \alpha \leq 90^\circ$ in intervals of $\left(\frac{1}{4}\right)^\circ$. All calculations were made to 3 sexagesimal (base 60) places.

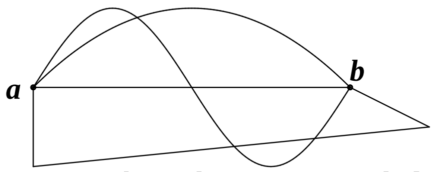
The etymology of the word *sine* is rather curious[42, pp 615-616]. The function we call sine was first given a name by Āryabhaṭa near the start of the sixth century AD. The name meant "half chord" and was later shortened to *jyā* meaning "chord". The Hindu word was translated into Arabic as *jîba*, which was a meaningless word phonetically derived from *jyā*, but (because the vowels in Arabic were not written) was written the same as *jaib*, which means bosom. When the Arabic was translated into Latin it became *sinus*. (*Jaib* means bosom, bay, or breast: *sinus* means bosom, bay or the fold of a toga around the breast.) The English word *sine* is derived from *sinus* phonetically.

9.31 Entertainment (Calculation of sines.) Design a computer program that will take as input a number x between 0 and .5, and will calculate $\sin(\pi x)$. (I choose $\sin(\pi x)$ instead of $\sin(x)$ so that you do *not* need to know the value of π to do this.)

9.2 Calculation of π

The proof of the next lemma depends on the following assumption, which is explicitly stated by Archimedes [2, page 3]. This assumption involves the ideas of *curve with given endpoints* and *length of curve* (which I will leave undiscussed).

9.32 Assumption. Let $\mathbf{a}$ and $\mathbf{b}$ be points in $\mathbf{R}^2$. Then of all curves with endpoints $\mathbf{a}$ and $\mathbf{b}$, the segment $[\mathbf{ab}]$ is the shortest.



curves with endpoints ***a*** and ***b***

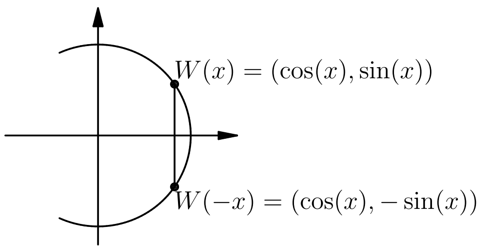
9.33 Lemma.

$$\sin(x) < x \text{ for all } x \in \mathbf{R}^+,$$

and

$$|\sin(x)| \le |x| \text{ for all } x \in \mathbf{R}.$$

Proof:
Case 1: Suppose $0 < x < \dfrac{\pi}{2}$.



Then (see the figure) the length of the arc joining $W(-x)$ to $W(x)$ in the first and fourth quadrants is $x + x = 2x$. (This follows from the definition of W.) The length of the segment $[W(x)W(-x)]$ is $2\sin(x)$. By our assumption, $2\sin(x) \leq 2x$, i.e., $\sin(x) \leq x$. Since both x and $\sin(x)$ are positive when $0 < x < \frac{\pi}{2}$, we also have $|\sin(x)| \leq |x|$.
Case 2: Suppose $x \geq \dfrac{\pi}{2}$. Then

$$\sin(x) \leq |\sin(x)| \leq 1 < \frac{\pi}{2} \leq x = |x|$$

so $\sin(x) \leq x$ and $|\sin(x)| \leq |x|$ in this case also. This proves the first assertion of lemma 9.33. If $x < 0$, then $-x > 0$, so

$$|\sin(x)| = |\sin(-x)| \leq |-x| = |x|.$$

Thus

$$|\sin(x)| \leq |x| \text{ for all } x \in \mathbf{R} \setminus \{0\},$$

and since the relation clearly holds when $x = 0$ the lemma is proved. |||

9.34 Lemma (Limits of sine and cosine.) *Let $a \in \mathbf{R}$. Let $\{a_n\}$ be a sequence in $\mathbf{R}$ such that $\{a_n\} \to a$. Then*

$$\{\cos(a_n)\} \to \cos(a) \text{ and } \{\sin(a_n)\} \to \sin(a).$$

Proof: By (9.25) we have

$$\cos(a_n) - \cos(a) = -2\sin\Big(\frac{a_n + a}{2}\Big)\sin\Big(\frac{a_n - a}{2}\Big),$$

so

$$\begin{aligned} 0 \leq |\cos(a_n) - \cos(a)| &\leq 2\Big|\sin\Big(\frac{a_n + a}{2}\Big)\Big|\Big|\sin\Big(\frac{a_n - a}{2}\Big)\Big| \\ &\leq 2\Big|\sin\Big(\frac{a_n - a}{2}\Big)\Big| \leq 2\Big|\frac{a_n - a}{2}\Big| = |a_n - a|. \end{aligned}$$

If $\{a_n\} \to a$, then $\{|a_n - a|\} \to 0$, so by the squeezing rule,

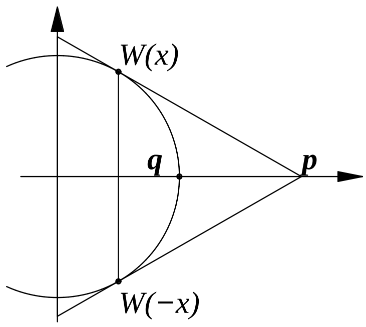
$$\{|\cos(a_n) - \cos(a)|\} \to 0.$$

This means that $\{\cos(a_n)\} \to \cos(a)$.

The proof that $\{\sin(a_n)\} \to \sin(a)$ is similar. |||

The proof of the next lemma involves another new assumption.

9.35 Assumption. Suppose $0 < x < \frac{\pi}{2}$. Let the tangent to the unit circle at $W(x)$ intersect the x axis at $\mathbf{p}$, and let $\mathbf{q} = (1, 0)$.



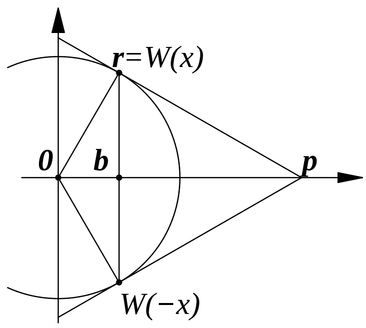
Then the circular arc joining $W(x)$ to $W(-x)$ (and passing through $\mathbf{q}$) is shorter than the curve made of the two segments $[W(x)\mathbf{p}]$ and $[\mathbf{p}W(-x)]$ (see the figure).

Remark: Archimedes makes a general assumption about curves that are *concave in the same direction* [2, pages 2-4] which allows him to prove our assumption.

9.36 Lemma. *If* $0 < x < \frac{\pi}{2}$, *then*

$$x \leq \frac{\sin(x)}{\cos(x)}.$$

Proof: Suppose $0 < x < \frac{\pi}{2}$. Draw the tangents to the unit circle at $W(x)$ and $W(-x)$ and let the point at which they intersect the x-axis be $\mathbf{p}$. (By symmetry both tangents intersect the x-axis at the same point.) Let $\mathbf{b}$ be the point where the segment $[W(x)W(-x)]$ intersects the x-axis, and let $\mathbf{r} = W(x)$. Triangles $\mathbf{0br}$ and $\mathbf{0rp}$ are similar since they are right triangles with a common acute angle.



Hence

$$\frac{\text{distance}(\mathbf{r},\mathbf{b})}{\text{distance}(\mathbf{0},\mathbf{b})} = \frac{\text{distance}(\mathbf{p},\mathbf{r})}{\text{distance}(\mathbf{0},\mathbf{r})}$$

i.e.,

$$\frac{\sin(x)}{\cos(x)} = \frac{\text{distance}(\mathbf{p},\mathbf{r})}{1}.$$

Now the length of the arc joining $W(x)$ to $W(-x)$ is $2x$, and the length of the broken line from $\mathbf{r}$ to $\mathbf{p}$ to $W(-x)$ is $2\Big(\text{distance}(\mathbf{p},\mathbf{r})\Big) = 2\dfrac{\sin(x)}{\cos(x)}$, so by assumption 9.35,

$$2x \leq 2\frac{\sin(x)}{\cos x}$$

i.e.,

$$x \leq \frac{\sin(x)}{\cos(x)}.$$

This proves our lemma. ‖

9.37 Theorem. *Let $\{x_n\}$ be any sequence such that $x_n \neq 0$ for all n, and $\{x_n\} \to 0$. Then*

$$\Big\{\frac{\sin(x_n)}{x_n}\Big\} \to 1. \tag{9.38}$$

Hence if $\sin(x_n) \neq 0$ *for all $n \in \mathbf{Z}^+$ we also have*

$$\Big\{\frac{x_n}{\sin(x_n)}\Big\} \to 1.$$

Proof: If $x \in (0, \frac{\pi}{2})$, then it follows from lemma(9.36) that $\cos(x) \leq \dfrac{\sin(x)}{x}$. Since

$$\cos(-x) = \cos(x) \text{ and } \frac{\sin(-x)}{-x} = \frac{\sin(x)}{x},$$

it follows that

$$\cos(x) \leq \frac{\sin(x)}{x} \text{ whenever } 0 < |x| < \frac{\pi}{2}.$$

Hence by lemma 9.33 we have

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1 \text{ whenever } 0 < |x| < \frac{\pi}{2}. \tag{9.39}$$

Let $\{x_n\}$ be a sequence for which $x_n \neq 0$ for all $n \in \mathbf{Z}^+$ and $\{x_n\} \to 0$. Then we can find a number $N \in \mathbf{Z}^+$ such that for all $n \in \mathbf{Z}_{\geq N}(|x_n| < \dfrac{\pi}{2})$. By (9.39)

$$n \in \mathbf{Z}_{\geq N} \implies \cos(x_n) \leq \frac{\sin(x_n)}{x_n} \leq 1.$$

By lemma 9.34, we know that $\{\cos(x_n)\} \to 1$, so by the squeezing rule $\left\{\dfrac{\sin(x_n)}{x_n}\right\} \to 1$. ‖

9.40 Example (Calculation of π.) Since $\left\{\dfrac{\pi}{n}\right\} \to 0$, it follows from (9.38) that

$$\lim\left\{\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}\right\} = 1$$

and hence that

$$\lim\left\{n \sin\left(\frac{\pi}{n}\right)\right\} = \pi.$$

This result can be used to find a good approximation to π. By the half-angle formula, we have

$$\sin^2\left(\frac{t}{2}\right) = \frac{1 - \cos t}{2} = \frac{1}{2}\left(1 - \sqrt{1 - \sin^2 t}\right)$$

for $0 \leq t \leq \dfrac{\pi}{2}$. Here I have used the fact that $\cos t \geq 0$ for $0 \leq t \leq \dfrac{\pi}{2}$. Also $\sin(\dfrac{\pi}{2}) = 1$ so

$$\sin^2(\frac{\pi}{4}) = \frac{1}{2}\left(1 - \sqrt{1 - \sin^2\frac{\pi}{2}}\right) = \frac{1}{2}\left(1 - \sqrt{0}\right) = \frac{1}{2}.$$

$$\sin^2(\frac{\pi}{8}) = \frac{1}{2}\Big(1-\sqrt{1-\sin^2\frac{\pi}{4}}\Big) = \frac{1}{2}\Big(1-\sqrt{1-\frac{1}{2}}\Big) = \frac{1}{2}\Big(1-\sqrt{\frac{1}{2}}\Big).$$

By repeated applications of this process I can find $\sin^2\left(\frac{\pi}{2^n}\right)$ for arbitrary n, and then find

$$2^n \sin\left(\frac{\pi}{2^n}\right)$$

which will be a good approximation to π.

I wrote a set of Maple routines to do the calculations above. The procedure `sinsq(n)` calculates $\sin^2\left(\frac{\pi}{2^n}\right)$ and the procedure `mypi(m)` calculates $2^m \sin\left(\frac{\pi}{2^m}\right)$. The "fi" (which is "if" spelled backwards) is Maple's way of ending an "if" statement. "Digits := 20" indicates that all calculations are done to 20 decimal digits accuracy. The command "evalf(Pi)" requests the decimal approximation to π to be printed.

```
> sinsq :=
>    n-> if n=1 then 1;
>        else .5*(1-sqrt(1 - sinsq(n-1)));
>        fi;
```

sinsq := proc(n) options operator,arrow; if n = 1 then 1 else .5 -.5*sqrt(1-sinsq(n-1)) fi end

```
> mypi := m -> 2^m*sqrt(sinsq(m));
```

$$mypi := m \to 2^m \operatorname{sqrt}(\operatorname{sinsq}(m))$$

```
> Digits := 20;
```

$$Digits := 20$$

```
> mypi(4);
```

3.1214451522580522853

```
> mypi(8);
```

3.1415138011443010542

```
> mypi(12);
```

3.1415923455701030907

```
> mypi(16);
```

3.1415926523835057093

```
> mypi(20);
                    3.1415926533473327481
> mypi(24);
                    3.1415922701132732445
> mypi(28);
                    3.1414977446171452114
> mypi(32);
                    3.1267833885746006944
> mypi(36);
                              0
> mypi(40);
                              0
> evalf(Pi);
                    3.1415926535897932385
```

9.41 Exercise. Examine the output of the program above. It appears that $\pi = 0$. This certainly is not right. What can I conclude about π from my computer program?

9.42 Exercise. Show that the number $n \sin\left(\frac{\pi}{n}\right)$ is the area of a regular $2n$-gon inscribed in the unit circle. Make any reasonable geometric assumptions, but explain your ideas clearly.

9.3 Integrals of the Trigonometric Functions

9.43 Theorem (Integral of cos**)** *Let $[a, b]$ be an interval in* **R**. *Then the cosine function is integrable on $[a, b]$, and*

$$\int_a^b \cos = \sin(b) - \sin(a).$$

Proof: Let $[a, b]$ be any interval in **R**. Then cos is piecewise monotonic on $[a, b]$ and hence is integrable. Let $P_n = \{x_0, x_1, \cdots, x_n\}$ be the regular partition of $[a, b]$ into n equal subintervals, and let

$$S_n = \left\{\frac{x_0 + x_1}{2}, \frac{x_1 + x_2}{2}, \cdots, \frac{x_{n-1} + x_n}{2}\right\}$$

be the sample for P_n consisting of the midpoints of the intervals of P_n.

Let $\Delta_n = \dfrac{b-a}{n}$ so that $x_i - x_{i-1} = \Delta_n$ and $\dfrac{x_{i-1}+x_i}{2} = x_{i-1} + \dfrac{\Delta_n}{2}$ for $1 \le i \le n$. Then

$$\begin{aligned}
\sum(\cos, P_n, S_n) &= \sum_{i=1}^{n} \cos\Big(x_{i-1} + \frac{\Delta_n}{2}\Big) \cdot \Delta_n \\
&= \Delta_n \sum_{i=1}^{n} \cos\Big(x_{i-1} + \frac{\Delta_n}{2}\Big).
\end{aligned}$$

Multiply both sides of this equation by $\sin\Big(\dfrac{\Delta_n}{2}\Big)$ and use the identity

$$\sin(t)\cos(s) = \frac{1}{2}[\sin(s+t) - \sin(s-t)]$$

to get

$$\begin{aligned}
\sin\Big(\frac{\Delta_n}{2}\Big)\sum(\cos, P_n, S_n) &= \Delta_n \sum_{i=1}^{n} \sin\Big(\frac{\Delta_n}{2}\Big)\cos\Big(x_{i-1} + \frac{\Delta_n}{2}\Big) \\
&= \Delta_n \sum_{i=1}^{n} \frac{1}{2}[\sin(x_{i-1}+\Delta_n) - \sin(x_{i-1})] \\
&= \frac{\Delta_n}{2} \sum_{i=1}^{n} \sin(x_i) - \sin(x_{i-1}) \\
&= \frac{\Delta_n}{2}\Big[\Big(\sin(x_n) - \sin(x_{n-1})\Big) + \Big(\sin(x_{n-1}) - \sin(x_{n-2})\Big) \\
&\quad + \cdots + \Big(\sin(x_1) - \sin(x_0)\Big)\Big] \\
&= \frac{\Delta_n}{2}[\sin(x_n) - \sin(x_0)] \\
&= \frac{\Delta_n}{2}\Big(\sin(b) - \sin(a)\Big).
\end{aligned}$$

Thus

$$\sum(\cos, P_n, S_n) = \frac{\Big(\frac{\Delta_n}{2}\Big)}{\sin\Big(\frac{\Delta_n}{2}\Big)}\Big(\sin(b) - \sin(a)\Big).$$

(By taking n large enough we can guarantee that $\dfrac{\Delta_n}{2} < \pi$, and then $\sin\Big(\dfrac{\Delta_n}{2}\Big) \ne 0$, so we haven't divided by 0.) Thus by theorem 9.37

$$\begin{aligned}
\int_a^b \cos &= \lim\{\sum(\cos, P_n, S_n)\} \\
&= \lim\left\{\Big(\sin(b)-\sin(a)\Big)\Big(\frac{\frac{\Delta_n}{2}}{\sin\left(\frac{\Delta_n}{2}\right)}\Big)\right\}. \\
&= \Big(\sin(b)-\sin(a)\Big)\cdot\lim\left\{\Big(\frac{\frac{\Delta_n}{2}}{\sin\left(\frac{\Delta_n}{2}\right)}\Big)\right\} \\
&= \Big(\sin(b)-\sin(a)\Big)\cdot 1 = \sin(b)-\sin(a). \ \|
\end{aligned}$$

9.44 Exercise. Let $[a, b]$ be an interval in **R**. Show that

$$\int_a^b \sin = \cos(a) - \cos(b). \tag{9.45}$$

The proof is similar to the proof of (9.43). The magic factor $\sin\left(\frac{\Delta_n}{2}\right)$ is the same as in that proof.

9.46 Notation ($\int_b^a f$.) If f is integrable on the interval $[a, b]$, we define

$$\int_b^a f = -\int_a^b f \text{ or } \int_b^a f(t)dt = -\int_a^b f(t)dt.$$

This is a natural generalization of the convention for $A_b^a f$ in definition 5.67.

9.47 Theorem (Integrals of sin **and** cos.**)** *Let a and b be any real numbers. Then*

$$\int_a^b \cos = \sin(b) - \sin(a).$$

and

$$\int_a^b \sin = \cos(a) - \cos(b).$$

Proof: We will prove the first formula. The proof of the second is similar. If $a \le b$ then the conclusion follows from theorem 9.43.

If $b < a$ then

$$\int_a^b \cos = -\int_b^a \cos = -[\sin(a) - \sin(b)] = \sin(b) - \sin(a),$$

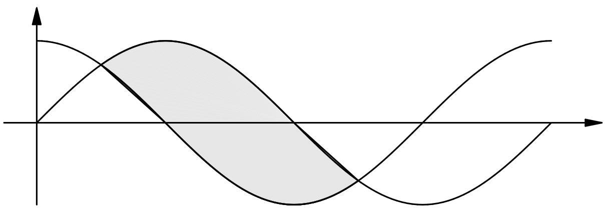
so the conclusion follows in all cases. ‖

9.48 Exercise. Find the area of the set

$$S_0^{\pi}(\sin) = \{(x,y) \colon 0 \le x \le \pi \text{ and } 0 \le y \le \sin x\}.$$

Draw a picture of $S_0^{\pi}(\sin)$.

9.49 Exercise. Find the area of the shaded figure, which is bounded by the graphs of the sine and cosine functions.



9.50 Example. By the change of scale theorem we have for $a < b$ and $c > 0$.

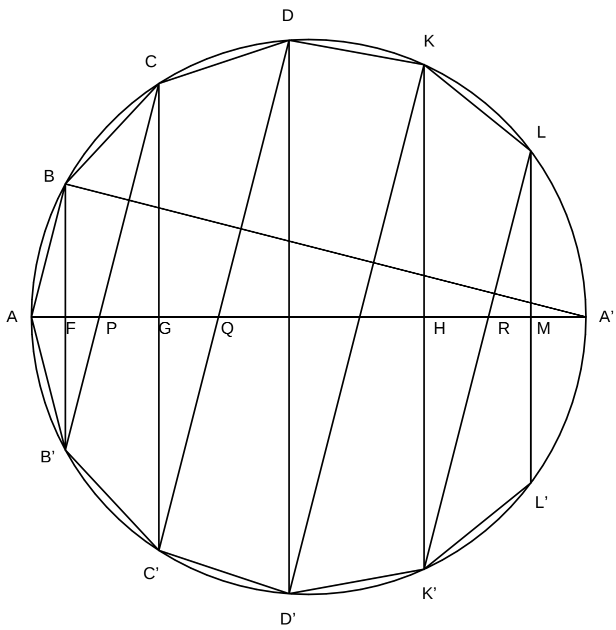
$$\begin{aligned}\int_a^b \sin(cx)dx &= \frac{1}{c}\int_{ca}^{cb} \sin x\, dx \\ &= \frac{-\cos(cb) + \cos(ca)}{c}\end{aligned}$$

$$\begin{aligned}\int_a^b \cos(cx)dx &= \frac{1}{c}\int_{ca}^{cb} \cos x\, dx \\ &= \frac{\sin(cb) - \sin(ca)}{c}\end{aligned}$$

9.51 Entertainment (Archimedes sine integral) In *On the Sphere and Cylinder 1.*, Archimedes states the following proposition: (see figure on next page)

Statement A:

> If a polygon be inscribed in a segment of a circle LAL' so that all its sides excluding the base are equal and their number even, as $LK \ldots A \ldots K'L'$, A being the middle point of segment, and if the lines BB', CC',... parallel to the base LL' and joining pairs of angular points be drawn, then
>
> $$(BB' + CC' + \ldots + LM) : AM = A'B : BA,$$
>
> where M is the middle point of LL' and AA' is the diameter through M.[2, page 29]



We will now show that this result can be reformulated in modern notation as follows.
Statement B: Let ϕ be a number in $[0, \pi]$, and let n be a positive integer. Then there exists a partitition-sample sequence $(\{P_n\}, \{S_n\})$ for $[0, \phi]$, such that

$$\sum(\sin, P_n, S_n) = (1 - \cos(\phi))\frac{\phi}{2n+1}\,\frac{\cos(\frac{\phi}{2n})}{\sin(\frac{\phi}{2n})}. \tag{9.52}$$

In exercise (9.56) you are asked to show that (9.52) implies that

$$\int_0^{\phi} \sin = 1 - \cos(\phi).$$

Proof that statement A implies statement B: Assume that statement A is true. Take the circle to have radius equal to 1, and let

$$\begin{aligned} \phi &= \text{length of arc}(AL) \\ \frac{\phi}{n} &= \text{length of arc}(AB). \end{aligned}$$

Then

$$BB' + CC' + \ldots + LM = 2\sin(\frac{\phi}{n}) + 2\sin(\frac{2\phi}{n}) + \cdots + 2\sin(\frac{(n-1)\phi}{n}) + \sin(\phi),$$

and

$$AM = 1 - \cos(\phi).$$

Let

$$P_n = \{0, \frac{2\phi}{2n+1}, \frac{4\phi}{2n+1}, \cdots, \frac{2n\phi}{2n+1}, \phi\},$$

and

$$S_n = \{0, \frac{\phi}{n}, \frac{2\phi}{n}, \cdots, \frac{n\phi}{n}\}.$$

Then P_n is a partition of $[0, \phi]$ with mesh equal to $\frac{2\phi}{2n+1}$, and S_n is a sample for P_n, so $(\{P_n\}, \{S_n\})$ is a partition-sample sequence for $[0, \phi]$, and we have

$$\sum(\sin, P_n, S_n) = \frac{2\phi}{2n+1}\Big(\sin(\frac{\phi}{n}) + \sin(\frac{2\phi}{n}) + \cdots + \sin(\frac{(n-1)\phi}{n}) + \frac{1}{2}\sin(\phi)\Big).$$

By Archimedes' formula, we conclude that

$$\sum(\sin, P_n, S_n) = (1 - \cos(\phi))\frac{\phi}{2n+1} \cdot \frac{A'B}{BA}. \tag{9.53}$$

We have

$$\begin{aligned} \text{length arc}(BA) &= \frac{\phi}{n}, \\ \text{length arc}(BA') &= \pi - \frac{\phi}{n}. \end{aligned}$$

By the formula for the length of a chord (9.30) we have

$$\frac{A'B}{BA} = \frac{\text{chord}(AB')}{\text{chord}(BA)} = \frac{2\sin(\frac{\text{arc}(AB')}{2})}{2\sin(\frac{\text{arc}(BA)}{2})} = \frac{\sin(\frac{(\pi - \frac{\phi}{n})}{2}))}{\sin(\frac{(\frac{\phi}{n})}{2})} = \frac{\cos(\frac{\phi}{2n})}{\sin(\frac{\phi}{2n})} \tag{9.54}$$

Equation (9.52) follows from (9.53) and (9.54).

Prove statement A above. Note that (see the figure from statement A)

$$AM = AF + FP + PG + GQ + \cdots + HR + RM, \tag{9.55}$$

and each summand on the right side of (9.55) is a side of a right triangle similar to triangle $A'BA$.

9.56 Exercise. Assuming equation (9.52), show that

$$\int_0^\phi \sin = 1 - \cos(\phi).$$

9.4 Indefinite Integrals

9.57 Theorem. *Let a, b, c be real numbers. If f is a function that is integrable on each interval with endpoints in $\{a, b, c\}$ then*

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

Proof: The case where $a \leq b \leq c$ is proved in theorem 8.18. The rest of the proof is exactly like the proof of exercise 5.69. |||

9.58 Exercise. Prove theorem 9.57.

We have proved the following formulas:

$$\begin{aligned}
\int_a^b x^r dx &= \frac{b^{r+1} - a^{r+1}}{r+1} \text{ for } 0 < a < b \quad r \in \mathbf{Q} \setminus \{-1\}, && (9.59)\\
\int_a^b \frac{1}{t} dt &= \ln(b) - \ln(a) \text{ for } 0 < a < b, \\
\int_a^b \sin(ct)dt &= \frac{-\cos(cb) + \cos(ca)}{c} \text{ for } a < b, \text{ and } c > 0, \\
\int_a^b \cos(ct)dt &= \frac{\sin(cb) - \sin(ca)}{c} \text{ for } a < b, \text{ and } c > 0. && (9.60)
\end{aligned}$$

In each case we have a formula of the form

$$\int_a^b f(t)dt = F(b) - F(a).$$

This is a general sort of situation, as is shown by the following theorem.

9.61 Theorem (Existence of indefinite integrals.) *Let J be an interval in $\mathbf{R}$, and let $f: J \to \mathbf{R}$ be a function such that f is integrable on every subinterval $[p, q]$ of J. Then there is a function $F: J \to \mathbf{R}$ such that for all $a, b \in J$*

$$\int_a^b f(t)dt = F(b) - F(a).$$

Proof: Choose a point $c \in J$ and define

$$F(x) = \int_c^x f(t)dt \text{ for all } x \in J.$$

Then for any points a, b in J we have

$$F(b) - F(a) = \int_c^b f(t)dt - \int_c^a f(t)dt = \int_a^b f(t)dt.$$

We've used the fact that

$$\int_c^b f(t)dt = \int_c^a f(t)dt + \int_a^b f(t)dt \text{ for all } a, b, c \in J. \parallel$$

9.62 Definition (Indefinite integral.) Let f be a function that is integrable on every subinterval of an interval J. An *indefinite integral for f on J* is any function $F\colon J \to \mathbf{R}$ such that $\int_a^b f(t)dt = F(b) - F(a)$ for all $a, b \in J$.

A function that has an indefinite integral always has infinitely many indefinite integrals, since if F is an indefinite integral for f then so is $F + c$ for any number c:

$$(F + c)(b) - (F + c)(a) = (F(b) + c) - (F(a) + c) = F(b) - F(a).$$

The following notation is used for indefinite integrals. One writes $\int f(t)dt$ to denote an indefinite integral for f. The t here is a dummy variable and can be replaced by any available symbol. Thus, based on formulas (9.59) - (9.60), we write

$$\begin{aligned}
\int x^r dx &= \frac{x^{r+1}}{r+1} \text{ if } r \in \mathbf{Q} \setminus \{-1\} \\
\int \frac{1}{t} dt &= \ln(t) \\
\int \sin(ct)dt &= -\frac{\cos(ct)}{c} \text{ if } c > 0 \\
\int \cos(ct)dt &= \frac{\sin(ct)}{c} \text{ if } c > 0.
\end{aligned}$$

We might also write

$$\int x^r dr = \frac{x^{r+1}}{r+1} + 3.$$

Some books always include an arbitrary constant with indefinite integrals, e.g.,

$$\int x^r dr = \frac{x^{r+1}}{r+1} + C \text{ if } r \in \mathbf{Q} \setminus \{-1\}.$$

The notation for indefinite integrals is treacherous. If you see the two equations

$$\int x^3 dx = \frac{1}{4}x^4$$

and

$$\int x^3 dx = \frac{1}{4}(x^4 + 1),$$

then you want to conclude

$$\frac{1}{4}x^4 = \frac{1}{4}(x^4 + 1), \tag{9.63}$$

which is wrong. It would be more logical to let the symbol $\int f(x)dx$ denote the set of *all* indefinite integrals for f. If you see the statements

$$\frac{1}{4}x^4 \in \int x^3 dx$$

and

$$\frac{1}{4}(x^4 + 1) \in \int x^3 dx,$$

you are not tempted to make the conclusion in (9.63).

9.64 Theorem (Sum theorem for indefinite integrals) *Let f and g be functions each of which is integrable on every subinterval of an interval J, and let $c, k \in \mathbf{R}$. Then*

$$\int \Big(cf(x) + kg(x)\Big)dx = c\int f(x)dx + k\int g(x)dx. \tag{9.65}$$

Proof: The statement (9.65) means that if F is an indefinite integral for f and G is an indefinite integral for G, then $cF + kG$ is an indefinite integral for $cf + kg$.

Let F be an indefinite integral for f and let G be an indefinite integral for g. Then for all $a, b \in J$

$$\begin{aligned}
\int_a^b \Big(cf(x) + kg(x)\Big)dx &= \int_a^b cf(x)dx + \int_a^b kg(x)dx \\
&= c\int_a^b f(x)dx + k\int_a^b g(x)dx \\
&= c\Big(F(b) - F(a)\Big) + k\Big(G(b) - G(a)\Big) \\
&= \Big(cF(b) + kG(b)\Big) - \Big(cF(a) + kG(a)\Big) \\
&= (cF + kG)(b) - (cF + kG)(a).
\end{aligned}$$

It follows that $cF + kG$ is an indefinite integral for $cf + kg$. |||

9.66 Notation ($F(t)\ |_a^b$.) If F is a function defined on an interval J, and if a, b are points in J we write $F(t)\ |_a^b$ for $F(b) - F(a)$. The t here is a dummy variable, and sometimes the notation is ambiguous, e.g. $x^2 - t^2\ |_0^1$. In such cases we may write $F(t)\ |_{t=a}^{t=b}$. Thus

$$(x^2 - t^2)\ |_{x=0}^{x=1} = (1 - t^2) - (0 - t^2) = 1$$

while

$$(x^2 - t^2)\ |_{t=0}^{t=1} = (x^2 - 1) - (x^2 - 0) = -1.$$

Sometimes we write $F\ |_a^b$ instead of $F(t)\ |_a^b$.

9.67 Example. It follows from our notation that if F is an indefinite integral for f on an interval J then

$$\int_a^b f(t)dt = F(t)\ |_a^b$$

and this notation is used as follows:

$$\begin{aligned}
\int_a^b 3x^2 dx &= x^3\Big|_a^b = b^3 - a^3. \\
\int_0^\pi \cos(x)dx &= \sin(x)\Big|_0^\pi = 0 - 0 = 0. \\
\int_0^\pi \sin(3x)dx &= \frac{-\cos 3x}{3}\Big|_0^\pi = \frac{-\cos(3\pi)}{3} + \frac{\cos(0)}{3} = \frac{2}{3}. \\
\int_0^2 (4x^2 + 3x + 1)dx &= 4\Big(\frac{x^3}{3}\Big) + 3\Big(\frac{x^2}{2}\Big) + x\Big|_0^2 \\
&= 4 \cdot \frac{8}{3} + 3 \cdot \frac{4}{2} + 2 = \frac{56}{3}.
\end{aligned}$$

In the last example I have implicitly used

$$\int (4x^2 + 3x + 1)dx = 4\int x^2 dx + 3\int x\ dx + \int 1\ dx.$$

9.68 Example. By using the trigonometric identities from theorem 9.21 we can calculate integrals of the form $\int_a^b \sin^n(cx)\cos^m(kx)dx$ where m, n are non-negative integers and $c, k \in \mathbf{R}$. We will find

$$\int_0^{\frac{\pi}{2}} \sin^3(x) \cdot \cos(3x)dx.$$

We have

$$\sin^2(x) = \frac{1-\cos(2x)}{2},$$

so

$$\begin{aligned}
\sin^3(x) &= \sin^2(x)\sin(x) = \frac{1}{2}\sin(x) - \frac{1}{2}\cos(2x)\sin(x) \\
&= \frac{1}{2}\sin(x) - \frac{1}{2}\cdot\frac{1}{2}\Big(\sin(3x) - \sin(x)\Big) \\
&= \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x).
\end{aligned}$$

Thus

$$\begin{aligned}
\sin^3(x)\cdot\cos(3x) &= \frac{3}{4}\cos(3x)\sin(x) - \frac{1}{4}\cos(3x)\sin(3x) \\
&= \frac{3}{8}[\sin(4x) - \sin(2x)] - \frac{1}{8}\sin(6x).
\end{aligned}$$

Hence

$$\int_0^{\pi/2} \sin^3(x)\cdot\cos(3x)\,dx$$

$$\begin{aligned}
&= \frac{3}{8}\frac{(-\cos(4x))}{4}\Bigg|_0^{\frac{\pi}{2}} - \frac{3}{8}\frac{(-\cos(2x))}{2}\Bigg|_0^{\frac{\pi}{2}} - \frac{1}{8}\frac{(-\cos(6x))}{6}\Bigg|_0^{\frac{\pi}{2}} \\
&= \frac{3}{32}\Big(-\cos(2\pi) + \cos(0)\Big) + \frac{3}{16}\Big(\cos(\pi) - \cos(0)\Big) + \frac{1}{48}\Big(\cos(3\pi) - \cos(0)\Big) \\
&= \frac{3}{16}(-1-1) + \frac{1}{48}(-1-1) = -\frac{3}{8} - \frac{1}{24} = \frac{-10}{24} = -\frac{5}{12}.
\end{aligned}$$

The method here is clear, but a lot of writing is involved, and there are many opportunities to make errors. In practice I wouldn't do a calculation of this sort by hand. The Maple command

```
> int((sin(x))^3*cos(3*x),x=0..Pi/2);
```

responds with the value

```
- 5/12
```

9.69 Exercise. Calculate the integrals

$$\int_0^{\frac{\pi}{2}} \sin x \, dx, \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \text{ and } \int_0^{\frac{\pi}{2}} \sin^4 x \, dx.$$

Then determine the values of

$$\int_0^{\frac{\pi}{2}} \cos x \, dx, \int_0^{\frac{\pi}{2}} \cos^2 x \, dx \text{ and } \int_0^{\frac{\pi}{2}} \cos^4 x \, dx$$

without doing any calculations. (But include an explanation of where your answer comes from.)

9.70 Exercise. Find the values of the following integrals. If the answer is geometrically clear then don't do any calculations, but explain why the answer is geometrically clear.

a) $\displaystyle\int_1^2 \frac{1}{x^3} dx$

b) $\displaystyle\int_{-1}^1 x^{11}(1+x^2)^3 \, dx$

c) $\displaystyle\int_0^2 \sqrt{4-x^2} \, dx$

d) $\displaystyle\int_0^\pi (x+\sin(2x))dx$

e) $\displaystyle\int_{-1}^1 \frac{1}{x^2} dx$

f) $\displaystyle\int_1^4 \frac{4+x}{x} dx$

g) $\displaystyle\int_0^1 \sqrt{x} dx$

h) $\displaystyle\int_1^2 \frac{4}{x} dx$

i) $\displaystyle\int_0^1 (1-2x)^2 dx$

j) $\displaystyle\int_0^1 (1-2x) dx$

k) $\int_0^{\pi} \sin(7x)\, dx$

l) $\int_0^{\pi} \sin(8x)\, dx$

9.71 Exercise.

$$\begin{aligned} \text{Let } A &= \int_0^{\pi/2} (\sin(4x))^5 dx \\ B &= \int_0^{\pi/2} (\sin(3x))^5 dx \\ C &= \int_0^{\frac{\pi}{2}} (\cos(3x))^5 dx. \end{aligned}$$

Arrange the numbers A, B, C in increasing order. Try to do the problem without making any explicit calculations. By making rough sketches of the graphs you should be able to come up with the answers. Sketch the graphs, and explain how you arrived at your conclusion. No "proof" is needed.