Chapter 4

Analytic Geometry

4.1 Addition of Points

From now on I will denote points in the plane by lower case boldface letters, e.g. $\mathbf{a}, \mathbf{b}, \cdots$. If I specify a point $\mathbf{a}$ and do not explicitly write down its components, you should assume $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$, $\cdots, \mathbf{k} = (k_1, k_2)$, etc. The one exception to this rule is that I will always take

$$\mathbf{x} = (x, y).$$

4.1 Definition (Addition of Points) If $\mathbf{a}$ and $\mathbf{b}$ are points in $\mathbf{R}^2$ and $t \in \mathbf{R}$, we define

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) \\
\mathbf{a} - \mathbf{b} &= (a_1, a_2) - (b_1, b_2) = (a_1 - b_1, a_2 - b_2) \\
t\mathbf{a} &= t(a_1, a_2) = (ta_1, ta_2).
\end{aligned}$$

If $t \neq 0$, we will write $\dfrac{\mathbf{a}}{t}$ for $\dfrac{1}{t}\mathbf{a}$; i.e., $\dfrac{\mathbf{a}}{t} = \left(\dfrac{a_1}{t}, \dfrac{b_1}{t}\right)$. We will abbreviate $(-1)\mathbf{a}$ by $-\mathbf{a}$, and we will write $\mathbf{0} = (0, 0)$.

4.2 Theorem. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be arbitrary points in* $\mathbf{R}^2$ *and let* s, t *be arbitrary numbers. Then we have:*

Addition is commutative,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

Addition is associative,

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

We have the following law that resembles the associative law for multiplication:

$$s(t\mathbf{a}) = (st)\mathbf{a}.$$

We have the following distributive laws:

$$(s + t)\mathbf{a} = s\mathbf{a} + t\mathbf{a}, \tag{4.3}$$

$$s(\mathbf{a} + \mathbf{b}) = s\mathbf{a} + s\mathbf{b}. \tag{4.4}$$

Also,

$$1\mathbf{a} = \mathbf{a}, \quad 0\mathbf{a} = \mathbf{0} \text{ and } \mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$$

All of these properties follow easily from the corresponding properties of real numbers. I will prove the commutative law and one of the distributive laws, and omit the remaining proofs.

Proof of Commutative Law: Let $\mathbf{a}$, $\mathbf{b}$ be points in $\mathbf{R}^2$. By the commutative law for $\mathbf{R}$,

$$a_1 + b_1 = b_1 + a_1 \text{ and } a_2 + b_2 = b_2 + a_2.$$

Hence

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) = (b_1 + a_1, b_2 + a_2) \\ &= (b_1, b_2) + (a_1, a_2) = \mathbf{b} + \mathbf{a}. \end{aligned}$$

and hence $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

Proof of (4.3): Let $s, t \in \mathbf{R}$ and let $\mathbf{a} \in \mathbf{R}^2$. By the distributive law for $\mathbf{R}$ we have

$$(s + t)a_1 = sa_1 + ta_1 \text{ and } (s + t)a_2 = sa_2 + ta_2.$$

Hence,

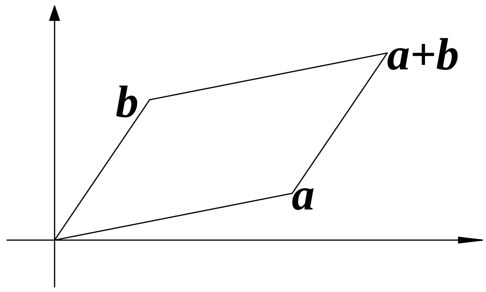
$$\begin{aligned} (s + t)\mathbf{a} &= (s + t)(a_1, a_2) = ((s + t)a_1, (s + t)a_2) = (sa_1 + ta_1, sa_2 + ta_2) \\ &= (sa_1, sa_2) + (ta_1, ta_2) = s(a_1, a_2) + t(a_1, a_2) \\ &= s\mathbf{a} + t\mathbf{a}, \end{aligned}$$

i.e,

$$(s+t)\mathbf{a} = s\mathbf{a} + t\mathbf{a}. \ \|$$

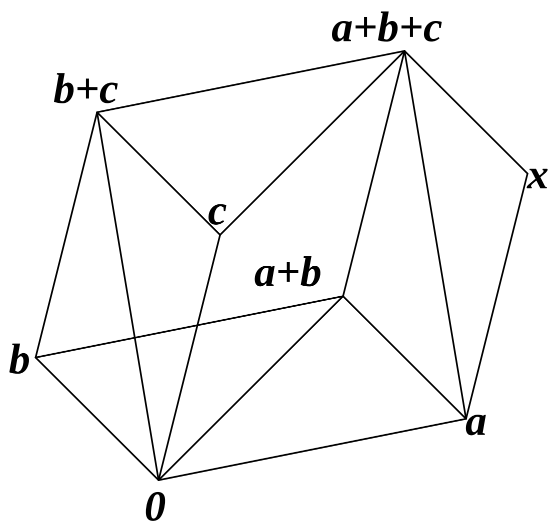
4.5 Notation (Lines in $\mathbf{R}^2$.) If $\mathbf{a}, \mathbf{b}$ are distinct points in $\mathbf{R}^2$, I will denote the (infinite) line through $\mathbf{a}$ and $\mathbf{b}$ by $\mathbf{ab}$, and I will denote the line segment joining $\mathbf{a}$ to $\mathbf{b}$ by $[\mathbf{ab}]$. Hence $[\mathbf{ab}] = [\mathbf{ba}]$.

Remark: Let $\mathbf{a}$, $\mathbf{b}$ be points in $\mathbf{R}^2$ such that $\mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$ are not all in a straight line. Then $\mathbf{a}+\mathbf{b}$ is the vertex opposite $\mathbf{0}$ in the parallelogram whose other three vertices are $\mathbf{b}$, $\mathbf{0}$ and $\mathbf{a}$.



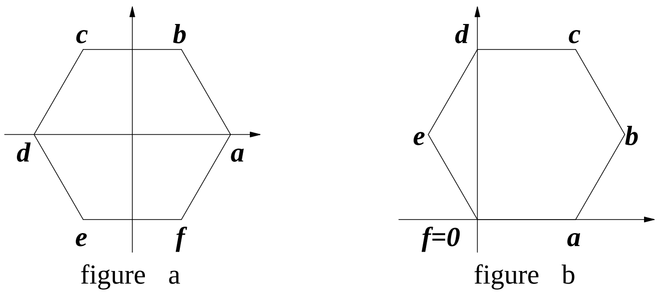
Proof: In this proof I will suppose $a_1 \neq 0$ and $b_1 \neq 0$, so that neither of $\mathbf{0a}, \mathbf{0b}$ is a vertical line. (I leave the other cases to you.) The slope of line $\mathbf{0a}$ is $\dfrac{a_2 - 0}{a_1 - 0} = \dfrac{a_2}{a_1}$, and the slope of $\mathbf{b}(\mathbf{a}+\mathbf{b})$ is $\dfrac{(a_2+b_2)-b_2}{(a_1+b_1)-b_1} = \dfrac{a_2}{a_1}$. Thus the lines $\mathbf{0a}$ and $\mathbf{b}(\mathbf{a}+\mathbf{b})$ are parallel.

The slope of line $\mathbf{0b}$ is $\dfrac{b_2 - 0}{b_1 - 0} = \dfrac{b_2}{b_1}$, and the slope of $\mathbf{a}(\mathbf{a}+\mathbf{b})$ is $\dfrac{(a_2+b_2)-a_2}{(a_1+b_1)-a_1} = \dfrac{b_2}{b_1}$. Thus the lines $\mathbf{0b}$ and $\mathbf{a}(\mathbf{a}+\mathbf{b})$ are parallel. It follows that the figure $\mathbf{0a}(\mathbf{a}+\mathbf{b})\mathbf{b}$ is a parallelogram, i.e., $\mathbf{a}+\mathbf{b}$ is the fourth vertex of a parallelogram having $\mathbf{0}$, $\mathbf{a}$, and $\mathbf{b}$ as its other vertices. $\|$



4.6 Example. In the figure you should be able to see the parallelograms defining $\mathbf{a}+\mathbf{b}$, $(\mathbf{a}+\mathbf{b})+\mathbf{c}$, $\mathbf{b}+\mathbf{c}$ and $\mathbf{a}+(\mathbf{b}+\mathbf{c})$. Also you should be able to see geometrically that $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$. What is the point marked $\mathbf{x}$ in the figure?

4.7 Exercise. In figure a), $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{f}$ are the vertices of a regular hexagon centered at $\mathbf{0}$. Sketch the points $\mathbf{a}+\mathbf{b}$, $(\mathbf{a}+\mathbf{b})+\mathbf{c}$, $(\mathbf{a}+\mathbf{b}+\mathbf{c})+\mathbf{d}$, $(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})+\mathbf{e}$, and $(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e})+\mathbf{f}$ as accurately as you can.



In figure b), $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{f}$ are the vertices of a regular hexagon with $\mathbf{f}=\mathbf{0}$. Sketch the points $\mathbf{a}+\mathbf{b}$, $(\mathbf{a}+\mathbf{b})+\mathbf{c}$, $(\mathbf{a}+\mathbf{b}+\mathbf{c})+\mathbf{d}$, and $(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})+\mathbf{e}$ as accurately as you can. (This problem should be done geometrically. Do not calculate the coordinates of any of these points.)

4.8 Example (Line segment) We will now give an analytical description for a non-vertical line segment $[\mathbf{ab}]$, $(a_1 \neq b_1)$. Suppose first that $a_1 < b_1$. The equation for the line through $\mathbf{a}$ and $\mathbf{b}$ is

$$y = a_2 + \frac{b_2 - a_2}{b_1 - a_1}(x - a_1).$$

Hence a point (x, y) is in $[\mathbf{ab}]$ if and only if there is a number $x \in [a_1, b_1]$ such that

$$\begin{aligned} (x, y) &= \Big(x, a_2 + \frac{b_2 - a_2}{b_1 - a_1}(x - a_1)\Big) \\ &= \Big(a_1 + (x - a_1), a_2 + \frac{b_2 - a_2}{b_1 - a_1}(x - a_1)\Big) \\ &= (a_1, a_2) + (x - a_1)(1, \frac{b_2 - a_2}{b_1 - a_1}) \\ &= \mathbf{a} + \frac{x - a_1}{b_1 - a_1}(b_1 - a_1, b_2 - a_2) \\ &= \mathbf{a} + \frac{x - a_1}{b_1 - a_1}(\mathbf{b} - \mathbf{a}). \end{aligned}$$

Now

$$\begin{aligned} x \in [a_1, b_1] &\iff a_1 \leq x \leq b_1 \\ &\iff 0 \leq x - a_1 \leq b_1 - a_1 \\ &\iff 0 \leq \frac{x - a_1}{b_1 - a_1} \leq 1. \end{aligned}$$

Thus

$$[\mathbf{ab}] = \{\mathbf{a} + t(\mathbf{b} - \mathbf{a}) : 0 \leq t \leq 1\}.$$

If $b_1 < a_1$ then

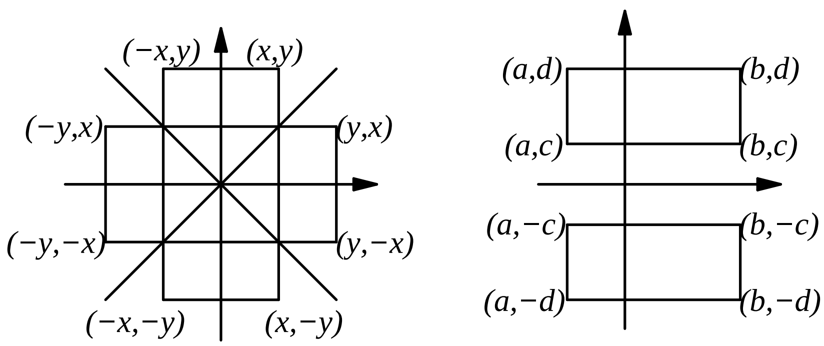
$$\begin{aligned} [\mathbf{ab}] &= [\mathbf{ba}] = \{\mathbf{b} + t(\mathbf{a} - \mathbf{b}) : 0 \leq t \leq 1\} \\ &= \{\mathbf{a} + (1 - t)(\mathbf{b} - \mathbf{a}) : 0 \leq t \leq 1\}. \end{aligned}$$

Now as t runs through all values in $[0, 1]$, we see that $1 - t$ also takes on all values in $[0, 1]$ so we get the same description for $[\mathbf{ab}]$ when $b_1 < a_1$ as we do when $a_1 < b_1$. Note that this description is exactly what you would expect from the pictures, and that it also works for vertical segments.

4.2 Reflections, Rotations and Translations

4.9 Definition (Reflections and Rotations.) We now define a family of functions from $\mathbf{R}^2$ to $\mathbf{R}^2$. If $(x, y) \in \mathbf{R}^2$, we define

$$
\begin{aligned}
I(x,y) &= (x,y) && \text{(Identity function.)} && (4.10)\\
H(x,y) &= (x,-y) && \text{(Reflection of } (x,y) \text{ about the horizontal axis.)}\\
V(x,y) &= (-x,y) && \text{(Reflection of } (x,y) \text{ about the vertical axis.)}\\
D_+(x,y) &= (y,x) && \text{(Reflection of } (x,y) \text{ about the line } y=x.\text{)}\\
D_-(x,y) &= (-y,-x) && \text{(Reflection of } (x,y) \text{ about the line } y=-x.\text{)}\\
R_{\pi/2}(x,y) &= (y,-x) && \text{(Clockwise rotation of } (x,y) \text{ by } \tfrac{\pi}{2}.\text{)}\\
R_{-\frac{\pi}{2}}(x,y) &= (-y,x) && \text{(Counter-clockwise rotation of } (x,y) \text{ by } \tfrac{\pi}{2}.\text{)}\\
R_{\pi}(x,y) &= (-x,-y) && \text{Rotation by } \pi. && (4.11)
\end{aligned}
$$



Each of the eight functions just defined carries every box to another box with the same area. You should be able to see from the picture that

$$H\left(B(a, b\colon c, d)\right) = B(a, b\colon -d, -c).$$

We can see this analytically as follows:

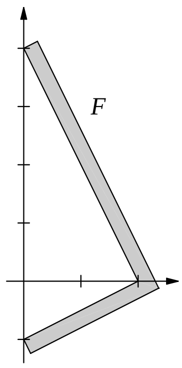
$$
\begin{aligned}
(x,y) \in B(a,b\colon c,d) &\iff a \le x \le b \text{ and } c \le y \le d\\
&\iff a \le x \le b \text{ and } -d \le -y \le -c\\
&\iff (x,-y) \in B(a,b\colon -d,-c)\\
&\iff H(x,y) \in B(a,b\colon -d,-c).
\end{aligned}
$$

I will usually omit the analytic justification in cases like this.

Each of the eight functions described in definition 4.9 carries the square $B(-1,1:-1,1)$ to itself.

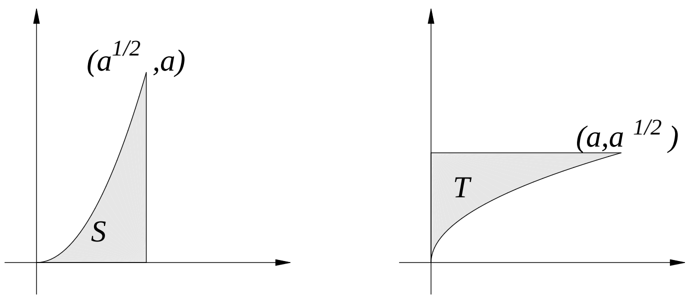
4.12 Definition (Symmetry of the square.) The eight functions defined in equations (4.10)-(4.11) are called *symmetries of the square.*

4.13 Exercise. Let F be the set shown in the figure. On one set of axes draw the sets $F, R_{\pi/2}(F), R_{-\frac{\pi}{2}}(F)$ and $R_{\pi}(F)$ (label the four sets). On another set of axes draw and label the sets $V(F), H(F), D_+(F)$ and $D_-(F)$.



4.14 Example. Let $a \in \mathbf{R}^+$ and let

$$\begin{aligned} S &= \{(x,y): 0 \le x \le \sqrt{a} \text{ and } 0 \le y \le x^2\} \\ T &= \{(x,y): 0 \le x \le a \text{ and } \sqrt{x} \le y \le a\}. \end{aligned}$$



From the picture it is clear that $D_+(S) = T$. An analytic proof of this result is as follows:

$$(x,y) \in S \iff 0 \le x \le \sqrt{a} \text{ and } 0 \le y \le x^2 \tag{4.15}$$

$$
\begin{aligned}
&\Longrightarrow \quad 0 \le y \le x^2 \le (\sqrt{a})^2 \text{ and } 0 \le \sqrt{y} \le x \le \sqrt{a} \\
&\Longrightarrow \quad 0 \le y \le a \text{ and } \sqrt{y} \le x \le \sqrt{a} \qquad (4.16) \\
&\Longleftrightarrow \quad (y, x) \in T \\
&\Longleftrightarrow \quad D_+(x, y) \in T.
\end{aligned}
$$

To show that $D_+(x,y) \in T \Longrightarrow (x,y) \in S$, I need to show that (4.16) implies (4.15). This follows because

$$0 \le y \le a \text{ and } \sqrt{y} \le x \le \sqrt{a} \Longrightarrow 0 \le x \le \sqrt{a} \text{ and } 0 \le y = (\sqrt{y})^2 \le x^2.$$

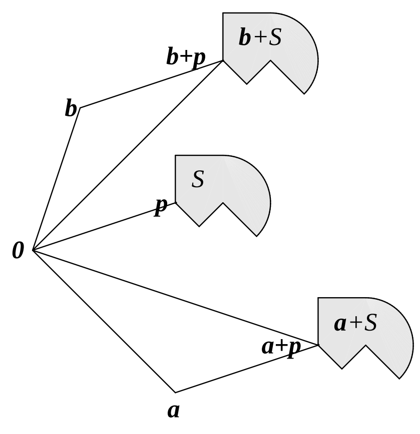
In exercise 2.18 you assumed that S and T have the same area. In general we will assume that if S is a set and F is a symmetry of the square, then S and $F(S)$ have the same area. (Cf. Assumption 5.11.)

4.17 Definition (Translate of a set.) Let S be a set in $\mathbf{R}^2$ and let $\mathbf{a} \in \mathbf{R}^2$. We define the set $\mathbf{a} + S$ by

$$\mathbf{a} + S = \{\mathbf{a} + \mathbf{s} \colon \mathbf{s} \in S\}.$$

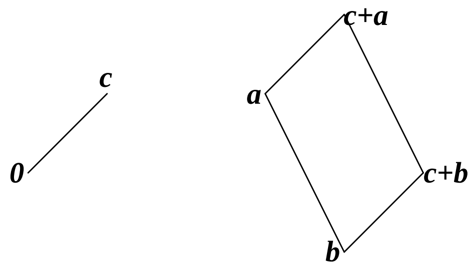
Sets of the form $\mathbf{a} + S$ will be called *translates of* S.

4.18 Example. The pictures below show some examples of translates. Intuitively each translate of S has the same shape as S and each translate of S has the same area as S.



4.19 Example (Translates of line segments.) Let $\mathbf{a}, \mathbf{b} \in \mathbf{R}^2$. If $\mathbf{c} \in \mathbf{R}^2$, then

$$\begin{aligned} \mathbf{c} + [\mathbf{ab}] &= \mathbf{c} + \{\mathbf{a} + t(\mathbf{b} - \mathbf{a}) : 0 \leq t \leq 1\} \\ &= \{\mathbf{c} + \mathbf{a} + t(\mathbf{b} - \mathbf{a}) : 0 \leq t \leq 1\} \\ &= \{\mathbf{c} + \mathbf{a} + t\,((\mathbf{c} + \mathbf{b}) - (\mathbf{c} + \mathbf{a})) : 0 \leq t \leq 1\} \\ &= [(\mathbf{c} + \mathbf{a})(\mathbf{c} + \mathbf{b})]. \end{aligned}$$



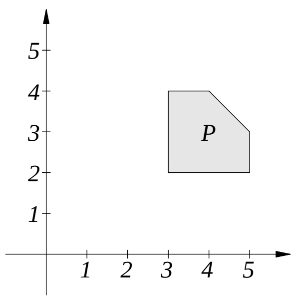
In particular $-\mathbf{a} + [\mathbf{a}, \mathbf{b}] = [\mathbf{0}, \mathbf{b} - \mathbf{a}]$, so any segment can be translated to a segment with $\mathbf{0}$ as an endpoint.

4.20 Exercise. Let a, b, c, d, r, s be real numbers with $a \leq b$ and $c \leq d$. Show that

$$(r, s) + B(a, b\colon c, d) = B(?, ?; ?, ?)$$

if the four question marks are replaced by suitable expressions. Include some explanation for your answer.

4.21 Exercise. Let P be the set shown in the figure below.



a) Sketch the sets $(-2, -2) + P$ and $(4, 1) + P$.

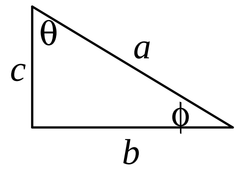
b) Sketch the sets $R_{\frac{\pi}{2}}((1, 1) + P)$ and $(1, 1) + R_{\frac{\pi}{2}}(P)$, where $R_{\frac{\pi}{2}}$ is defined as in definition 4.9

4.3 The Pythagorean Theorem and Distance.

Even though you are probably familiar with the Pythagorean theorem, the result is so important and non-obvious that I am including a proof of it.

4.22 Theorem (Pythagorean Theorem.) *In any right triangle, the square on the hypotenuse is equal to the sum of the squares on the two legs.*

Proof: Consider a right triangle T whose legs have length b and c, and whose hypotenuse has length a, and whose angles are ϕ and θ as shown in the figure.



We have $\phi + \theta = 90°$ since T is a right triangle.

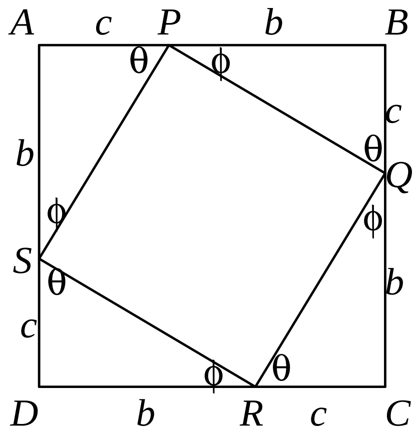


figure 1

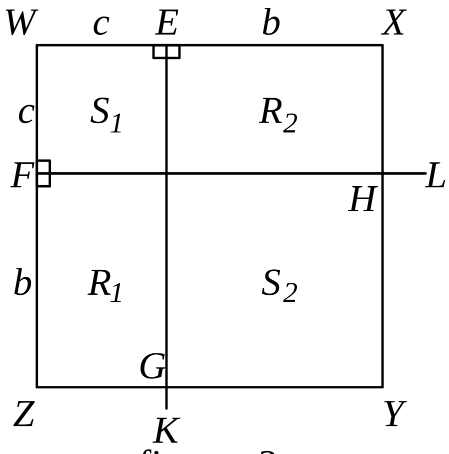


figure 2

Construct a square $ABCD$ with sides of length $b+c$, and find points P, Q, R, S dividing the sides of $ABCD$ into pieces of sizes b and c as shown in figure 1. Draw the lines PQ, QR, RS, and SP, thus creating four triangles congruent to T (i.e., four right triangles with legs of length b and c). Each angle of $PQRS$

is $180° - (\phi + \theta) = 180° - 90° = 90°$ so $PQRS$ is a square of side a. The four triangles in figure 1 each have area $\frac{1}{2}bc$, so

$$\text{area}(ABCD) - 4 \cdot \text{area}(T) = a^2 \tag{4.23}$$

or

$$(b + c)^2 - 2bc = a^2$$

and hence

$$b^2 + c^2 = a^2 \ \| \tag{4.24}$$

The proof just given uses a combination of algebra and geometry. I will now give a second proof that is completely geometrical.

Construct a second square $WXYZ$ with sides of length $b + c$, and mark off segments WE and WF of length c as shown in figure 2. Then draw EK perpendicular to WX and let EK intersect ZY at G, and draw FL perpendicular to WZ and let FL intersect XY at H. Then EGZ is a right angle, since the other angles of the quadrilateral $WEGZ$ are right angles. Similarly angle FHX is a right angle. Thus $WEGZ$ is a rectangle so $ZG = c$ and similarly $WFHX$ is a rectangle and $XH = c$. Moreover EG and FH are perpendicular since $EG \| WZ$ and $FH \| WX$. Thus the region labeled S_1 is a square with side c and the region labeled S_2 is a square with side b.

In figure 2 we have $\text{area}(R_1) = \text{area}(R_2) = 2\text{area}(T)$, and hence

$$\text{area}(WXYZ) - 4 \cdot \text{area}(T) = b^2 + c^2. \tag{4.25}$$

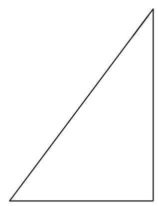
We have $\text{area}(ABCD) = \text{area}(WXYZ)$ since $ABCD$ and $WXYZ$ are both squares with side $b + c$. Hence from equations (4.23) and (4.25) we see that

$$a^2 = b^2 + c^2. \ \|$$

Although the theorem we just proved is named for Pythagoras (fl. 530–510 B.C) , it was probably known much earlier. There is evidence that it was known to the Babylonians circa 1000 BC[27, pp 118-121]. Legend has it that

> Emperor Yǔ[circa 21st century B.C.] quells floods, he deepens rivers and streams, observes the shape of mountains and valleys, surveys the high and low places, relieves the greatest calamities

> and saves the people from danger. He leads the floods east into the sea and ensures no flooding or drowning. This is made possible because of the Gōugŭ theorem ...[47, page 29].



Gōugŭ shape

"Gōugŭ" is the shape shown in the figure, and the Gōugŭ theorem is our Pythagorean theorem. The prose style here is similar to that of current day mathematicians trying to get congress to allocate funds for the support of mathematics.

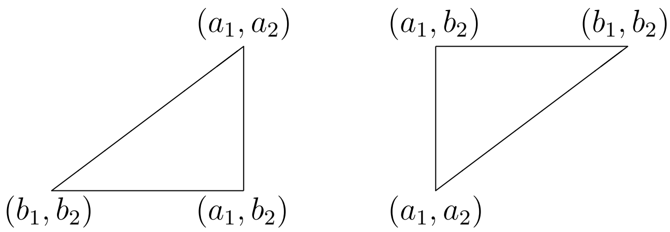
Katyayana(c. 600 BC or 500BC??) stated the general theorem:

> The rope [stretched along the length] of the diagonal of a rectangle makes an [area] which the vertical and horizontal sides make together.[27, page 229]

4.26 Theorem (Distance formula.) *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are points in* $\mathbf{R}^2$ *then the distance from* $\mathbf{a}$ *to* $\mathbf{b}$ *is*

$$d(\mathbf{a}, \mathbf{b}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

Proof: Draw the vertical line through $\mathbf{a}$ and the horizontal line through $\mathbf{b}$. These lines intersect at the point $\mathbf{p} = (a_1, b_2)$. The length of $[\mathbf{ap}]$ is $|a_2 - b_2|$ and the length of $[\mathbf{pb}]$ is $|a_1 - b_1|$ and $[\mathbf{ab}]$ is the hypotenuse of a right angle with legs $[\mathbf{ap}]$ and $[\mathbf{pb}]$.



By the Pythagorean theorem,

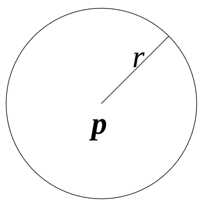
$$(\text{length}([\mathbf{ab}]))^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

so $\text{length}([\mathbf{a}, \mathbf{b}]) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$. |||

4.27 Notation ($d(\mathbf{a}, \mathbf{b})$, distance$(\mathbf{a}, \mathbf{b})$) If $\mathbf{a}$ and $\mathbf{b}$ are points in $\mathbf{R}^2$, I will denote the distance from $\mathbf{a}$ to $\mathbf{b}$ by either distance$(\mathbf{a}, \mathbf{b})$ or by $d(\mathbf{a}, \mathbf{b})$.

4.28 Definition (Circle.) Let $\mathbf{p} = (a, b)$ be a point in $\mathbf{R}^2$, and let $r \in \mathbf{R}^+$. The *circle with center* $\mathbf{p}$ *and radius* r is defined to be

$$\begin{aligned} C(\mathbf{p}, r) &= \{(x, y) \in \mathbf{R}^2 \colon d((x, y), (a, b)) = r\} \\ &= \{(x, y) \in \mathbf{R}^2 \colon \sqrt{(x - a)^2 + (y - b)^2} = r\} \\ &= \{(x, y) \in \mathbf{R}^2 \colon (x - a)^2 + (y - b)^2 = r^2\}. \end{aligned}$$



C(p,r)

The equation

$$(x - a)^2 + (y - b)^2 = r^2$$

is called the *equation of the circle* $C(\mathbf{p}, r)$. The circle $C((0, 0), 1)$ is called the *unit circle.*

We will now review the method for solving quadratic equations.

4.29 Theorem (Quadratic formula.) *Let A, B, and C be real numbers with $A \neq 0$.*

If $B^2 - 4AC < 0$, then the equation $Ax^2 + Bx + C = 0$ has no solutions in $\mathbf{R}$.

If $B^2 - 4AC \geq 0$, then the set of solutions of the equation $Ax^2 + Bx + C = 0$ is

$$\left\{ \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \right\}. \tag{4.30}$$

The set (4.30) contains one or two elements, depending on whether $B^2 - 4AC$ is zero or positive.)

Proof: Let A, B, C be real numbers with $A \neq 0$. Let $x \in \mathbf{R}$. Then

$$\begin{aligned} Ax^2 + Bx + C = 0 &\iff A\Big(x^2 + \frac{Bx}{A}\Big) = -C \\ &\iff A\Big(x^2 + \frac{Bx}{A} + \frac{B^2}{4A^2}\Big) = -C + \frac{AB^2}{4A^2} = -C + \frac{B^2}{4A} \\ &\iff A\Big(x + \frac{B}{2A}\Big)^2 = \frac{-4AC + B^2}{4A} \\ &\iff \Big(x + \frac{B}{2A}\Big)^2 = \frac{B^2 - 4AC}{4A^2} \end{aligned}$$

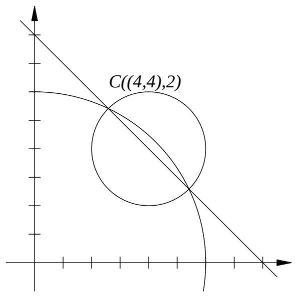
Hence $Ax^2+Bx+C = 0$ has no solutions unless $B^2-4AC \geq 0$. If $B^2-4AC \geq 0$, then the solutions are given by

$$\Big(x + \frac{B}{2A}\Big) = \frac{\pm\sqrt{B^2 - 4AC}}{2A}$$

i.e.,

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \quad \|$$

4.31 Example. Describe the set $C((0,0),6) \cap C((4,4),2)$.



The sketch suggests that this set will consist of two points in the first quadrant. Let (x, y) be a point in the intersection. Then

$$x^2 + y^2 = 36 \tag{4.32}$$

and

$$(x-4)^2 + (y-4)^2 = 4, \text{ i.e. } x^2 + y^2 - 8x - 8y + 28 = 0. \tag{4.33}$$

It follows that $36 - 8x - 8y + 28 = 0$, or $8x + 8y - 64 = 0$ or

$$y = 8 - x. \tag{4.34}$$

(The line whose equation is $y = 8 - x$ is shown in the figure. We've proved that the intersection is a subset of this line.) Replace y by $8 - x$ in equation (4.33) to obtain

$$x^2 + (8 - x)^2 - 8x - 8(8 - x) + 28 = 0$$

i.e.,

$$x^2 + 64 - 16x + x^2 - 8x - 64 + 8x + 28 = 0$$

i.e.,

$$2x^2 - 16x + 28 = 0$$

i.e.,

$$x^2 - 8x + 14 = 0.$$

By the quadratic formula, it follows that

$$x = \frac{8 \pm \sqrt{64 - 56}}{2} = \frac{8 \pm 2\sqrt{2}}{2} = 4 \pm \sqrt{2}.$$

By equation (4.34)

$$y = 8 - x = 4 \mp \sqrt{2}.$$

We have shown that if $(x, y) \in C((0, 0), 6) \cap C((4, 4), 2)$, then $(x, y) \in \{(4 + \sqrt{2}, 4 - \sqrt{2}), (4 - \sqrt{2}, 4 + \sqrt{2})\}$. It is easy to verify that each of the two calculated points satisfies both equations (4.32) and (4.33) so

$$C((0, 0), 6) \cap C((4, 4), 2) = \{(4 + \sqrt{2}, 4 - \sqrt{2}), (4 - \sqrt{2}, 4 + \sqrt{2})\}.$$