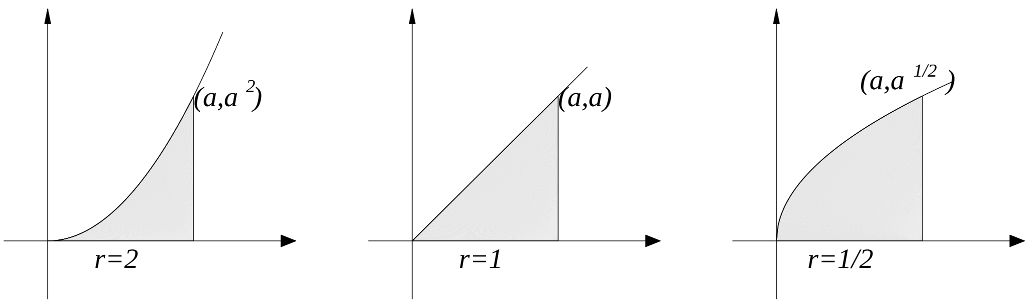
Chapter 2

Some Area Calculations

2.1 The Area Under a Power Function

Let a be a positive number, let r be a positive number, and let S_a^r be the set of points (x, y) in $\mathbf{R}^2$ such that $0 \le x \le a$ and $0 \le y \le x^r$. In this section we will begin an investigation of the area of S_a^r.



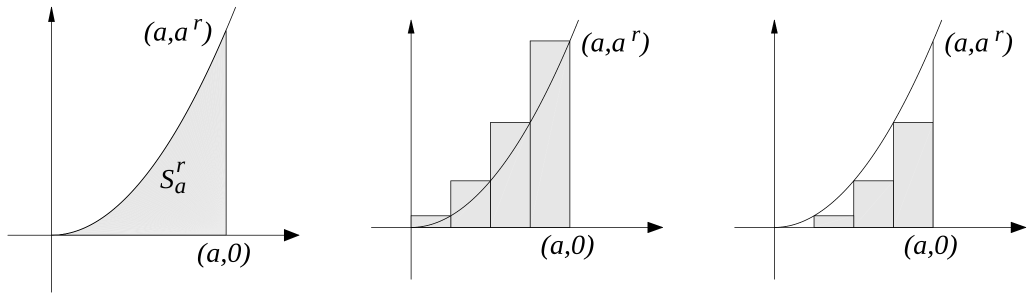
S_a^r for various positive values of r

Our discussion will not apply to negative values of r, since we make frequent use of the fact that for all non-negative numbers x and t

$$(x \le t) \text{ implies that } (x^r \le t^r).$$

Also 0^r is not defined when r is negative.

The figures for the argument given below are for the case $r = 2$, but you should observe that the proof does not depend on the pictures.



$S_a^r \subset \bigcup_{i=1}^4 O_i$ $\qquad$ $\bigcup_{i=1}^4 I_i \subset S_a^r$

Let n be a positive integer, and for $0 \leq i \leq n$, let $x_i = \dfrac{ia}{n}$.

Then $x_i - x_{i-1} = \dfrac{a}{n}$ for $1 \leq i \leq n$, so the points x_i divide the interval $[0, a]$ into n equal subintervals. For $1 \leq i \leq n$, let

$$\begin{aligned} I_i &= B(x_{i-1}, x_i\colon 0, x_{i-1}^r) \\ O_i &= B(x_{i-1}, x_i\colon 0, x_i^r). \end{aligned}$$

If $(x, y) \in S_a^r$, then $x_{i-1} \leq x \leq x_i$ for some index i, and $0 \leq y \leq x^r \leq x_i^r$, so

$$(x, y) \in B(x_{i-1},\ x_i\colon 0,\ x_i^r) = O_i \text{ for some } i \in \{1, \cdots, n\}.$$

Hence we have

$$S_a^r \subset \bigcup_{i=1}^n O_i,$$

and thus

$$\text{area}(S_a^r) \leq \text{area}(\bigcup_{i=1}^n O_i). \tag{2.1}$$

If $(x, y) \in I_i$, then $0 \leq x_{i-1} \leq x \leq x_i \leq a$ and $0 \leq y \leq x_{i-1}^r \leq x^r$ so $(x, y) \in S_a^r$. Hence, $I_i \subset S_a^r$ for all i, and hence

$$\bigcup_{i=1}^n I_i \subset S_a^r,$$

so that

$$\text{area}(\bigcup_{i=1}^{n} I_i) \leq \text{area}(S_a^r). \tag{2.2}$$

Now

$$\begin{aligned}\text{area}(I_i) &= \text{area}\left(B(x_{i-1}, x_i\colon 0, x_{i-1}^r)\right) \\ &= (x_i - x_{i-1})x_{i-1}^r = \frac{a}{n}\left(\frac{(i-1)a}{n}\right)^r = \frac{a^{r+1}}{n^{r+1}}(i-1)^r,\end{aligned}$$

and

$$\begin{aligned}\text{area}(O_i) &= \text{area}\left(B(x_{i-1}, x_i\colon 0, x_i^r)\right) \\ &= (x_i - x_{i-1})x_i^r = \frac{a}{n}\left(\frac{ia}{n}\right)^r = \frac{a^{r+1}}{n^{r+1}}i^r.\end{aligned}$$

Since the boxes I_i intersect only along their boundaries, we have

$$\begin{aligned}\text{area}(\bigcup_{i=1}^{n} I_i) &= \text{area}(I_1) + \text{area}(I_2) + \cdots + \text{area}(I_n) \\ &= \frac{a^{r+1}}{n^{r+1}}0^r + \frac{a^{r+1}}{n^{r+1}}1^r + \cdots + \frac{a^{r+1}}{n^{r+1}}(n-1)^r \\ &= \frac{a^{r+1}}{n^{r+1}}(1^r + 2^r + \cdots + (n-1)^r), \qquad (2.3)\end{aligned}$$

and similarly

$$\begin{aligned}\text{area}(\bigcup_{i=1}^{n} O_i) &= \text{area}(O_1) + \text{area}(O_2) + \cdots + \text{area}(O_n) \\ &= \frac{a^{r+1}}{n^{r+1}}1^r + \frac{a^{r+1}}{n^{r+1}}2^r + \cdots + \frac{a^{r+1}}{n^{r+1}}n^r \\ &= \frac{a^{r+1}}{n^{r+1}}(1^r + 2^r + \cdots + n^r).\end{aligned}$$

Thus it follows from equations (2.1) and (2.2) that

$$\frac{a^{r+1}}{n^{r+1}}(1^r + 2^r + \cdots + (n-1)^r) \leq \text{area}(S_a^r) \leq \frac{a^{r+1}}{n^{r+1}}(1^r + 2^r + \cdots + n^r). \tag{2.4}$$

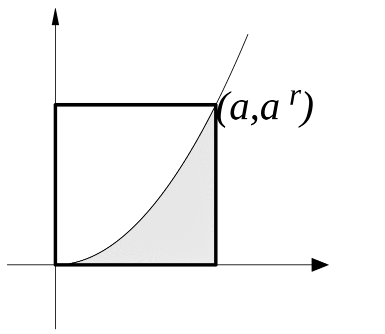
The geometrical question of finding the area of S_a^r has led us to the numerical problem of finding the sum

$$1^r + 2^r + \cdots + n^r.$$

We will study this problem in the next section.

2.5 Definition (Circumscribed box.) Let $\operatorname{cir}(S_a^r)$ be the smallest box containing (S_a^r). i.e.

$$\operatorname{cir}(S_a^r) = B(0, a; 0, a^r) \qquad (r \geq 0).$$



Notice that $\operatorname{area}(\operatorname{cir}(S_a^r)) = a \cdot a^r = a^{r+1}$. Thus equation (2.4) can be written as

$$\frac{(1^r + 2^r + \cdots + (n-1)^r)}{n^{r+1}} \leq \frac{\operatorname{area}(S_a^r)}{\operatorname{area}(\operatorname{cir}(S_a^r))} \leq \frac{(1^r + 2^r + \cdots + n^r)}{n^{r+1}}. \qquad (2.6)$$

Observe that the outside terms in (2.6) do not depend on a.

Now we will specialize to the case when $r = 2$. A direct calculation shows that

$$\begin{aligned} 1^2 &= 1, \\ 1^2 + 2^2 &= 5, \\ 1^2 + 2^2 + 3^2 &= 14, \\ 1^2 + 2^2 + 3^2 + 4^2 &= 30, \\ 1^2 + 2^2 + 3^2 + 4^2 + 5^2 &= 55. \end{aligned} \qquad (2.7)$$

There is a simple (?) formula for $1^2+2^2+\cdots+n^2$, but it is not particularly easy to guess this formula on the basis of these calculations. With the help of my computer, I checked that

$$1^2+\cdots+10^2=385 \text{ so } \frac{1^2+\cdots+10^2}{10^3}=.385$$

$$1^2+\cdots+100^2=338350 \text{ so } \frac{1^2+\cdots+100^2}{100^3}=.33835$$

$$1^2+\cdots+1000^2=333833500 \text{ so } \frac{1^2+\cdots+1000^2}{1000^3}=.3338335$$

Also

$$\begin{aligned}\frac{1^2+\cdots+999^2}{1000^3} &= \frac{1^2+\cdots+1000^2}{1000^3}-\frac{1000^2}{1000^3}=.3338335-.001\\ &= .3328335.\end{aligned}$$

Thus by taking $n=1000$ in equation (2.6) we see that

$$.332 \le \frac{\text{area}(S_a^2)}{\text{area}(\text{cir}(S_a^2))} \le .3339.$$

On the basis of the computer evidence it is very tempting to guess that

$$\text{area}(S_a^2)=\frac{1}{3}\text{area}(\text{cir}(S_a^2))=\frac{1}{3}a^3.$$

2.2 Some Summation Formulas

We will now develop a formula for the sum

$$1+2+\cdots+n.$$

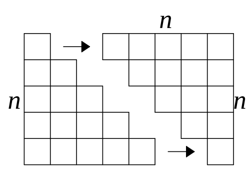


figure a

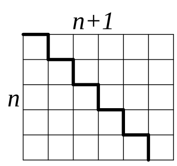


figure b

Figure (a) shows two polygons, each having area $1+2+\cdots+n$. If we slide the two polygons so that they touch, we create a rectangle as in figure (b) whose area is $n(n+1)$. Thus

$$2(1+2+\cdots+n)=n(n+1)$$

i.e.,

$$1+2+\cdots+n=\frac{n(n+1)}{2}. \tag{2.8}$$

The proof just given is quite attractive, and a proof similar to this was probably known to the Pythagoreans in the 6th or 5th centuries B.C. Cf [29, page 30]. The formula itself was known to the Babylonians much earlier than this[45, page 77], but we have no idea how they discovered it.

The idea here is special, and does not generalize to give a formula for $1^2+2^2+\cdots+n^2$. (A nice geometrical proof of the formula for the sum of the first n squares can be found in *Proofs Without Words* by Roger Nelsen[37, page 77], but it is different enough from the one just given that I would not call it a "generalization".) We will now give a second proof of (2.8) that generalizes to give formulas for $1^p+2^p+\cdots+n^p$ for positive integers p. The idea we use was introduced by Blaise Pascal [6, page 197] circa 1654.

For any real number k, we have

$$(k+1)^2-k^2=k^2+2k+1-k^2=2k+1.$$

Hence

$$\begin{aligned}
1^2-0^2 &= 2\cdot 0+1,\\
2^2-1^2 &= 2\cdot 1+1,\\
3^2-2^2 &= 2\cdot 2+1,\\
&\vdots\\
(n+1)^2-n^2 &= 2\cdot n+1.
\end{aligned}$$

Add the left sides of these $(n+1)$ equations together, and equate the result to the sum of the right sides:

$$(n+1)^2-n^2+\cdots+3^2-2^2+2^2-1^2+1^2-0^2=2\cdot(1+\cdots+n)+(n+1).$$

In the left side of this equation all of the terms except the first cancel. Thus

$$(n+1)^2=2(1+2+\cdots+n)+(n+1)$$

so

$$2(1+2+\cdots+n) = (n+1)^2 - (n+1) = (n+1)(n+1-1) = (n+1)n$$

and

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

This completes the second proof of (2.8).

To find $1^2+2^2+\cdots+n^2$ we use the same sort of argument. For any real number k we have

$$(k+1)^3 - k^3 = k^3 + 3k^2 + 3k + 1 - k^3 = 3k^2 + 3k + 1.$$

Hence,

$$\begin{aligned}
1^3 - 0^3 &= 3\cdot 0^2 + 3\cdot 0 + 1,\\
2^3 - 1^3 &= 3\cdot 1^2 + 3\cdot 1 + 1,\\
3^3 - 2^3 &= 3\cdot 2^2 + 3\cdot 2 + 1,\\
&\vdots\\
(n+1)^3 - n^3 &= 3\cdot n^2 + 3\cdot n + 1.
\end{aligned}$$

Next we equate the sum of the left sides of these $n+1$ equations with the sum of the right sides. As before, most of the terms on the left side cancel and we obtain

$$(n+1)^3 = 3(1^2+2^2+\cdots+n^2) + 3(1+2+\cdots+n) + (n+1).$$

We now use the known formula for $1+2+3+\cdots+n$:

$$(n+1)^3 = 3(1^2+2^2+\cdots+n^2) + \frac{3}{2}n(n+1) + (n+1)$$

so

$$\begin{aligned}
3(1^2+2^2+\cdots+n^2) &= (n+1)^3 - \frac{3}{2}n(n+1) - (n+1)\\
&= (n+1)\left((n+1)^2 - \frac{3}{2}n - 1\right)\\
&= (n+1)(n^2+2n+1-\frac{3}{2}n-1)\\
&= (n+1)\left(n^2+\frac{1}{2}n\right) = (n+1)n\left(n+\frac{1}{2}\right)\\
&= \frac{n(n+1)(2n+1)}{2},
\end{aligned}$$

and finally

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2.9}$$

You should check that this formula agrees with the calculations made in (2.7). The argument we just gave can be used to find formulas for $1^3 + 2^3 + \cdots + n^3$, and for sums of higher powers, but it takes a certain amount of stamina to carry out the details. To find $1^3 + 2^3 + \cdots + n^3$, you could begin with

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1 \text{ for all } k \in \mathbf{R}.$$

Add together the results of this equation for $k = 0, 1, \cdots, n$ and get

$$(n+1)^4 = 4(1^3 + 2^3 + \cdots + n^3) + 6(1^2 + 2^2 + \cdots + n^2) + 4(1 + \cdots + n) + (n+1).$$

Then use equations (2.8) and (2.9) to eliminate $1^2+2^2+\cdots+n^2$ and $1+\cdots+n$, and solve for $1^3 + 2^3 + \cdots + n^3$.

2.10 Exercise. A Complete the argument started above, and find the formula for $1^3 + 2^3 + \cdots + n^3$.

Jacob Bernoulli (1654–1705) considered the general formula for power sums. By using a technique similar to, but slightly different from Pascal's, he constructed the table below. Here $f(1) + f(2) + \cdots f(n)$ is denoted by $\int f(n)$, and $*$ denotes a missing term: Thus the $*$ in the fourth line of the table below indicates that there is no n^2 term, i.e. the coefficient of n^2 is zero.

> Thus we can step by step reach higher and higher powers and with slight effort form the following table.

Sums of Powers

$$
\begin{array}{lllllll}
\int n & = \frac{1}{2}nn & +\frac{1}{2}n, & & & & \\
\int nn & = \frac{1}{3}n^3 & +\frac{1}{2}nn & +\frac{1}{6}n, & & & \\
\int n^3 & = \frac{1}{4}n^4 & +\frac{1}{2}n^3 & +\frac{1}{4}nn, & & & \\
\int n^4 & = \frac{1}{5}n^5 & +\frac{1}{2}n^4 & +\frac{1}{3}n^3 & *-\frac{1}{30}n, & & \\
\int n^5 & = \frac{1}{6}n^6 & +\frac{1}{2}n^5 & +\frac{5}{12}n^4 & *-\frac{1}{12}nn, & & \\
\int n^6 & = \frac{1}{7}n^7 & +\frac{1}{2}n^6 & +\frac{1}{2}n^5 & *-\frac{1}{6}n^3 & *+\frac{1}{42}n, & \\
\int n^7 & = \frac{1}{8}n^8 & +\frac{1}{2}n^7 & +\frac{7}{12}n^6 & *-\frac{7}{24}n^4 & *+\frac{1}{12}nn, & \\
\int n^8 & = \frac{1}{9}n^9 & +\frac{1}{2}n^8 & +\frac{2}{3}n^7 & *-\frac{7}{15}n^5 & *+\frac{2}{9}n^3 & *-\frac{1}{30}n, \\
\int n^9 & = \frac{1}{10}n^{10} & +\frac{1}{2}n^9 & +\frac{3}{4}n^8 & *-\frac{7}{10}n^6 & *+\frac{1}{2}n^4 & *-\frac{3}{20}nn, \\
\int n^{10} & = \frac{1}{11}n^{11} & +\frac{1}{2}n^{10} & +\frac{5}{6}n^9 & *-1n^7 & *+1n^5 & *-\frac{1}{2}n^3 *+\frac{5}{66}n.
\end{array}
$$

> Whoever will examine the series as to their regularity may be able to continue the table[9, pages 317–320]. [1]

He then states a rule for continuing the table. The rule is not quite an explicit formula, rather it tells how to compute the next line easily when the previous lines are known.

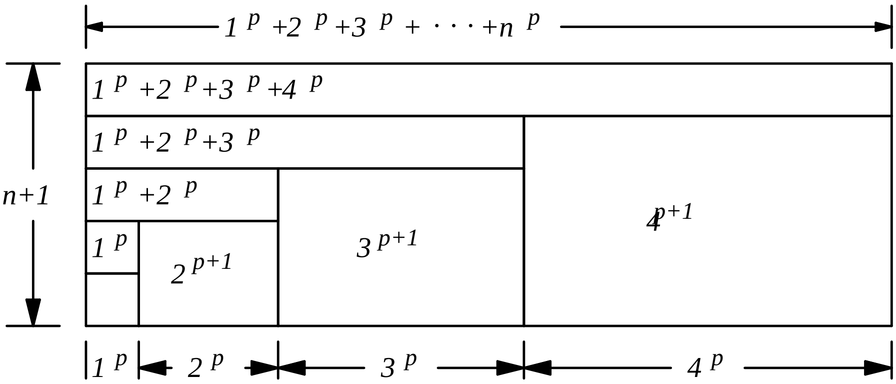
2.11 Entertainment (Bernoulli's problem.) Guess a way to continue the table. Your answer should be explicit enough so that you can actually calculate the next two lines of the table.

A formula for $1^2 + 2^2 + \cdots + n^2$ was proved by Archimedes (287-212 B.C.). (See Archimedes *On Conoids and Spheroids* in [2, pages 107-109]). The formula was known to the Babylonians[45, page 77] much earlier than this in the form

$$1^2 + 2^2 + \cdots + n^2 = (\frac{1}{3} + n \cdot \frac{2}{3})(1 + 2 + \cdots + n).$$

A technique for calculating general power sums has been known since circa 1000 A.D. At about this time Ibn-al-Haitham, gave a method based on the picture below, and used it to calculate the power sums up to $1^4 + 2^4 + \cdots + n^4$. The method is discussed in [6, pages 66–69]

[1] A typographical error in Bernoulli's table has been corrected here.



2.3 The Area Under a Parabola

If S_a^2 is the set of points (x, y) in $\mathbf{R}^2$ such that $0 \le x \le a$ and $0 \le y \le x^2$, then we showed in (2.6) that

$$\frac{1^2 + 2^2 + \cdots + (n-1)^2}{n^3} \le \frac{\text{area}(S_a^2)}{\text{area}(\text{cir}(S_a^2))} \le \frac{1^2 + \cdots + n^2}{n^3}.$$

By (2.9)

$$\begin{aligned}\frac{1^2 + 2^2 + \cdots + n^2}{n^3} &= \frac{n(n+1)(2n+1)}{n^3 \cdot 6} = \frac{1}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{2n}\right) \\ &= \frac{1}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right).\end{aligned}$$

Also

$$\begin{aligned}\frac{1^2 + 2^2 + \cdots + (n-1)^2}{n^3} &= \frac{(n-1)n((2(n-1)+1)}{n^3 \cdot 6} = \frac{1}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{2n}\right) \\ &= \frac{1}{3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right), \end{aligned} \tag{2.12}$$

so

$$\frac{1}{3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right) \le \frac{\text{area}(S_a^2)}{\text{area}(\text{cir}(S_a^2))} \le \frac{1}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right) \tag{2.13}$$

for all $n \in \mathbf{Z}^+$.

The right side of (2.13) is greater than $\frac{1}{3}$ and the left side is less than $\frac{1}{3}$ for all $n \in \mathbf{Z}^+$, but by taking n large enough, both sides can be made as close to $\frac{1}{3}$ as we please. Hence we conclude that the ratio $\frac{\text{area}(S_a^2)}{\text{area}(\text{cir}(S_a^2))}$ is equal to $\frac{1}{3}$. Thus, we have proved the following theorem:

2.14 Theorem (Area Under a Parabola.) *Let a be a positive real number and let S_a^2 be the set of points (x, y) in $\mathbf{R}^2$ such that $0 \leq x \leq a$ and $0 \leq y \leq x^2$. Then*

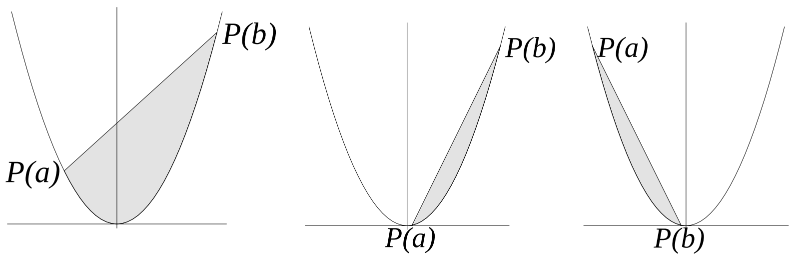
$$\frac{\text{area}(S_a^2)}{\text{area(box circumscribed about } S_a^2)} = \frac{1}{3},$$

i.e.

$$\text{area}(S_a^2) = \frac{1}{3}a^3.$$

Remark: The last paragraph of the proof of theorem 2.14 is a little bit vague. How large is "large enough" and what does "as close as we please" mean? Archimedes and Euclid would not have considered such an argument to be a proof. We will reconsider the end of this proof after we have developed the language to complete it more carefully. (Cf Example 6.51.)

The first person to calculate the area of a parabolic segment was Archimedes (287-212 B.C.). The parabolic segment considered by Archimedes corresponds to the set $S(a, b)$ bounded by the parabola $y = x^2$ and the line joining $P(a) = (a, a^2)$ to $P(b) = (b, b^2)$ where $(a < b)$.

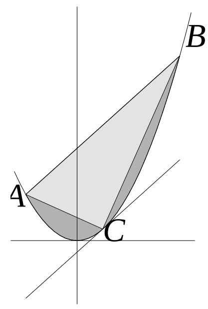


Parabolic Segments

2.15 Exercise. Show that the area of the parabolic segment $S(a, b)$ just described is $\frac{(b-a)^3}{6}$. Use theorem 2.14 and any results from Euclidean geometry that you need. You may assume that $0 < a \leq b$. The cases where $a < 0 < b$ and $a < b < 0$ are all handled by similar arguments.

The result of this exercise shows that the area of a parabolic segment depends only on its width. Thus the segment determined by the points $(-1, 1)$ and $(1, 1)$ has the same area as the segment determined by the points $(99, 9801)$ and $(101, 10201)$, even though the second segment is 400 times as tall as the first, and both segments have the same width. Does this seem reasonable?

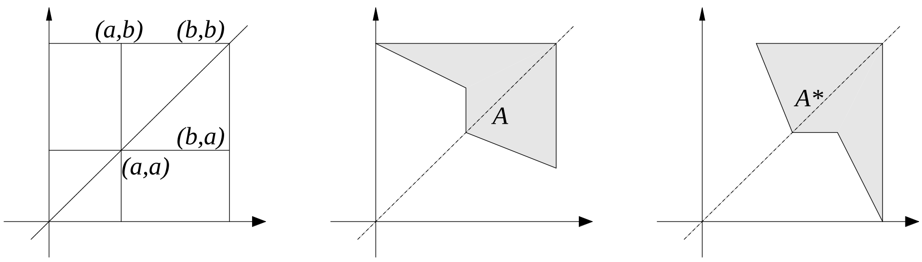
Remark: Archimedes stated his result about the area of a parabolic segment as follows. The area of the parabolic segment cut off by the line AB is four thirds of the area of the inscribed triangle ABC, where C is the point on the parabola at which the tangent line is parallel to AB. We cannot verify Archimedes formula at this time, because we do not know how to find the point C.



2.16 Exercise. Verify Archimedes' formula as stated in the above remark for the parabolic segment $S(-a, a)$. In this case you can use your intuition to find the tangent line.

The following definition is introduced as a hint for exercise 2.18A

2.17 Definition (Reflection about the line $y = x$) If S is a subset of $\mathbf{R}^2$, then the *reflection of S about the line $y = x$* is defined to be the set of all points (x, y) such that $(y, x) \in S$.

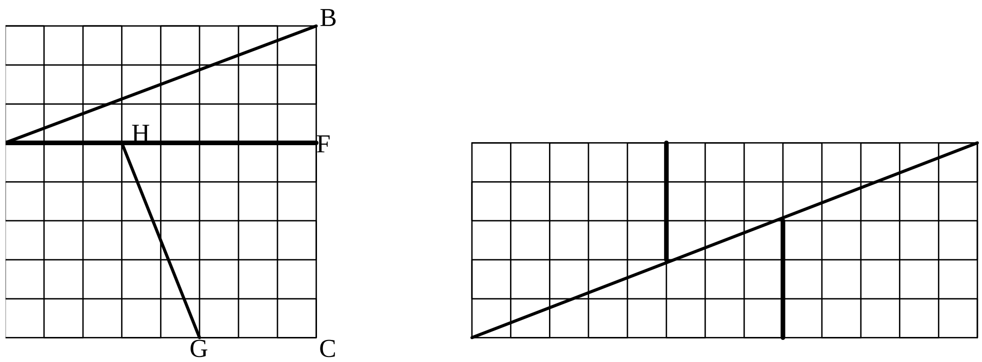


A^* is the reflection of A about the line $y = x$

If S^* denotes the reflection of S about the line $y = x$, then S and S^* have the same area.

2.18 Exercise. A Let $a \in \mathbf{R}^+$ and let T_a be the set of all points (x, y) such that $0 \leq x \leq a$ and $0 \leq y \leq \sqrt{x}$. Sketch the set T_a and find its area.

2.19 Exercise. In the first figure below, the 8×8 square $ABCD$ has been divided into two 3×8 triangles and two trapezoids by means of the lines EF, EB and GH. In the second figure the four pieces have been rearranged to form an 5×13 rectangle. The square has area 64 , and the rectangle has area 65. Where did the extra unit of area come from? (This problem was taken from W. W. Rouse Ball's *Mathematical Recreations* [4, page 35]. Ball says that the earliest reference he could find for the problem is 1868.)



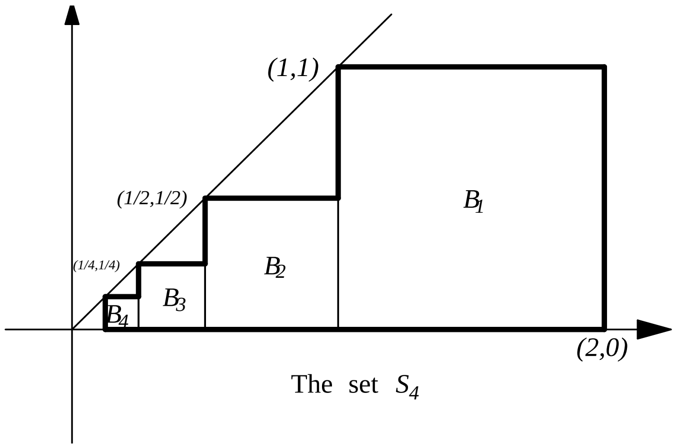
2.4 Finite Geometric Series

For each n in $\mathbf{Z}^+$ let B_n denote the box

$$B_n = B\left(\frac{1}{2^{n-1}}, \frac{2}{2^{n-1}} : 0, \frac{1}{2^{n-1}}\right),$$

and let

$$S_n = B_1 \cup B_2 \cup \cdots \cup B_n = \bigcup_{j=1}^{n} B_j.$$



The set S_4

I want to find the area of S_n. I have

$$\text{area}(B_n) = \left(\frac{2}{2^{n-1}} - \frac{1}{2^{n-1}}\right) \cdot \left(\frac{1}{2^{n-1}} - 0\right) = \frac{1}{2^{n-1}} \cdot \frac{1}{2^{n-1}} = \frac{1}{4^{n-1}}.$$

Since the boxes B_i intersect only along their boundaries, we have

$$\begin{aligned} \text{area}(S_n) &= \text{area}(B_1) + \text{area}(B_2) + \cdots + \text{area}(B_n) \\ &= 1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-1}}. \end{aligned} \tag{2.20}$$

Thus

$$\begin{aligned} \text{area}(S_1) &= 1, \\ \text{area}(S_2) &= 1 + \frac{1}{4} = \frac{5}{4}, \\ \text{area}(S_3) &= \frac{5}{4} + \frac{1}{16} = \frac{20}{16} + \frac{1}{16} = \frac{21}{16} = \frac{21}{4^2}, \\ \text{area}(S_4) &= \frac{21}{16} + \frac{1}{64} = \frac{84}{64} + \frac{1}{64} = \frac{85}{64} = \frac{85}{4^3}. \end{aligned} \tag{2.21}$$

You probably do not see any pattern in the numerators of these fractions, but in fact area(S_n) is given by a simple formula, which we will now derive.

2.22 Theorem (Finite Geometric Series.) *Let r be a real number such that $r \neq 1$. Then for all $n \in \mathbf{Z}^+$*

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}. \tag{2.23}$$

Proof: Let

$$S = 1 + r + r^2 + \cdots + r^{n-1}.$$

Then

$$rS = r + r^2 + \cdots + r^{n-1} + r^n.$$

Subtract the second equation from the first to get

$$S(1 - r) = 1 - r^n,$$

and thus

$$S = \frac{1 - r^n}{1 - r}. \ \|\!|$$ [2]

Remark: Theorem 2.22 is very important, and you should remember it. Some people find it easier to remember the proof than to remember the formula. It would be good to remember both.

If we let $r = \frac{1}{4}$ in (2.23), then from equation (2.20) we obtain

$$\begin{aligned} \text{area}(S_n) &= 1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-1}} \\ &= \frac{1 - \frac{1}{4^n}}{1 - \frac{1}{4}} = \frac{4}{3}\left(1 - \frac{1}{4^n}\right) \\ &= \frac{4^n - 1}{3 \cdot 4^{n-1}}. \end{aligned} \tag{2.24}$$

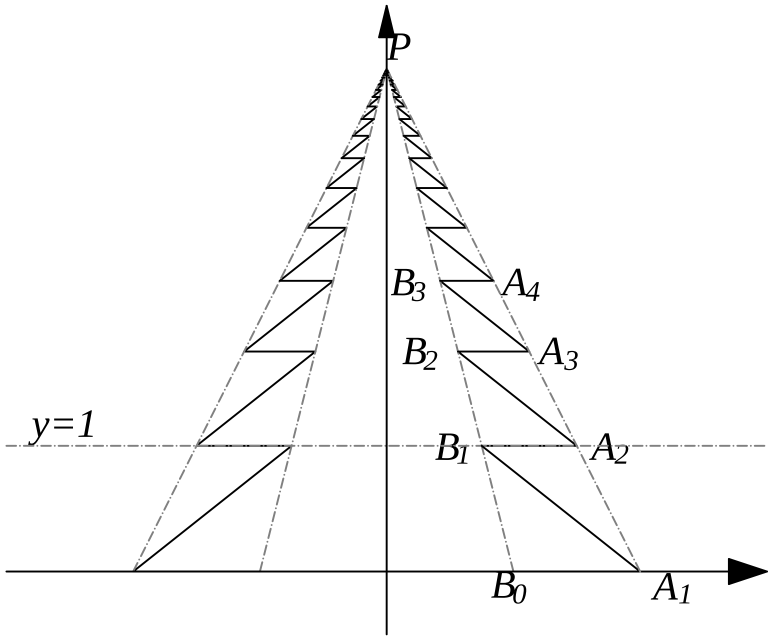
As a special case, we have

$$\text{area}(S_4) = \frac{4^4 - 1}{3 \cdot 4^3} = \frac{256 - 1}{3 \cdot 4^3} = \frac{255}{3 \cdot 4^3} = \frac{85}{4^3}$$

which agrees with equation (2.21).

[2] We use the symbol $\|\!|$ to denote the end of a proof.

2.25 Entertainment (Pine Tree Problem.) Let T be the subset of $\mathbf{R}^2$ sketched below:



Here $P = (0, 4)$, $B_0 = (1, 0)$, $A_1 = (2, 0)$, and B_1 is the point where the line B_0P intersects the line $y = 1$. All of the points A_j lie on the line PA_1, and all of the points B_j lie on the line PB_0. All of the segments A_iB_{i-1} are horizontal, and all segments A_jB_j are parallel to A_1B_1. Show that the area of T is $\frac{44}{7}$. You will probably need to use the formula for a geometric series.

2.26 Exercise.
(a) Find the number

$$1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \cdots + \frac{1}{7^{100}}$$

accurate to 8 decimal places.
(b) Find the number

$$1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \cdots + \frac{1}{7^{1000}}$$

accurate to 8 decimal places.

(You may use a calculator, but you can probably do this without using a calculator.)

2.27 Exercise. A Let

$$\begin{aligned} a_1 &= .027 \\ a_2 &= .027027 \\ a_3 &= .027027027 \\ &etc. \end{aligned}$$

Use the formula for a finite geometric series to get a simple formula for a_n. What rational number should the infinite decimal $.027027027\cdots$ represent? Note that

$$a_3 = .027(1.001001) = .027(1 + \frac{1}{1000} + \frac{1}{1000^2}).$$

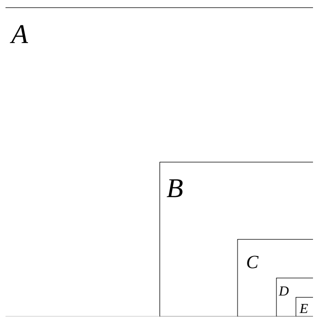
The Babylonians[45, page 77] knew that

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^n + (2^n - 1), \tag{2.28}$$

i.e. they knew the formula for a finite geometric series when $r = 2$.

Euclid knew a version of the formula for a finite geometric series in the case where r is a positive integer.

Archimedes knew the sum of the finite geometric series when $r = \frac{1}{4}$. The idea of Archimedes' proof is illustrated in the figure.



If the large square has side equal to 2, then

$$\begin{aligned} A &&= A &= 3 \\ \tfrac{1}{4}A &&= B & \\ (\tfrac{1}{4})^2 A &= \tfrac{1}{4}B &= C & \\ (\tfrac{1}{4})^3 A &= \tfrac{1}{4}C &= D. & \end{aligned}$$

Hence

$$\begin{aligned}(1+\tfrac{1}{4}+(\tfrac{1}{4})^2+(\tfrac{1}{4})^3)A &= (A+B+C+D)=4-E\\ &= 4-(\tfrac{1}{8})^2=4-(\tfrac{1}{4})^3=4(1-(\tfrac{1}{4})^4).\end{aligned}$$

i.e.

$$(1+\tfrac{1}{4}+(\tfrac{1}{4})^2+(\tfrac{1}{4})^3)\cdot 3=4(1-(\tfrac{1}{4})^4).$$

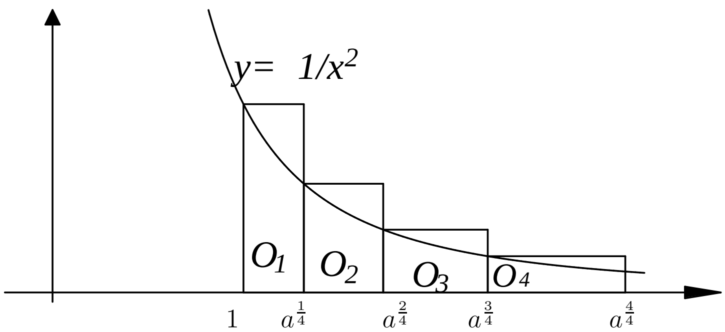
For the details of Archimedes' argument see [2, pages 249-250].

2.29 Exercise. Explain why formula (2.28) is a special case of the formula for a finite geometric series.

2.5 Area Under the Curve $y=\frac{1}{x^2}$

The following argument is due to Pierre de Fermat (1601-1665) [19, pages 219-222]. Later we will use Fermat's method to find the area under the curve $y=x^\alpha$ for all α in $\mathbf{R}\setminus\{-1\}$.

Let a be a real number with $a>1$, and let S_a be the set of points (x,y) in $\mathbf{R}^2$ such that $1\le x\le a$ and $0\le y\le \dfrac{1}{x^2}$. I want to find the area of S_a.



Let n be a positive integer. Note that since $a>1$, we have

$$1<a^{\frac{1}{n}}<a^{\frac{2}{n}}<\cdots<a^{\frac{n}{n}}=a.$$

Let O_j be the box

$$O_j = B\left(a^{\frac{j-1}{n}}, a^{\frac{j}{n}} \colon 0, \frac{1}{\left(a^{\frac{j-1}{n}}\right)^2}\right).$$

Thus the upper left corner of O_j lies on the curve $y = \dfrac{1}{x^2}$.
To simplify the notation, I will write

$$b = a^{\frac{1}{n}}.$$

Then

$$O_j = B\left(b^{j-1}, b^j : 0, \frac{1}{b^{2(j-1)}}\right),$$

and

$$\text{area}(O_j) = \frac{b^j - b^{j-1}}{b^{2(j-1)}} = \frac{(b-1)b^{j-1}}{b^{2(j-1)}} = \frac{(b-1)}{b^{(j-1)}}.$$

Hence

$$\begin{aligned}
\text{area}\left(\bigcup_{j=1}^{n} O_j\right) &= \text{area}(O_1) + \text{area}(O_2) + \cdots + \text{area}(O_n) \\
&= (b-1) + \frac{(b-1)}{b} + \cdots + \frac{(b-1)}{b^{(n-1)}} \\
&= (b-1)\left(1 + \frac{1}{b} + \cdots + \frac{1}{b^{(n-1)}}\right).
\end{aligned}$$

Observe that we have here a finite geometric series, so

$$\text{area}\left(\bigcup_{j=1}^{n} O_j\right) = (b-1)\left(\frac{1 - \frac{1}{b^n}}{1 - \frac{1}{b}}\right) \tag{2.30}$$

$$= b\left(1 - \frac{1}{b}\right)\left(\frac{1 - \frac{1}{b^n}}{1 - \frac{1}{b}}\right) = b\left(1 - \frac{1}{b^n}\right). \tag{2.31}$$

Now

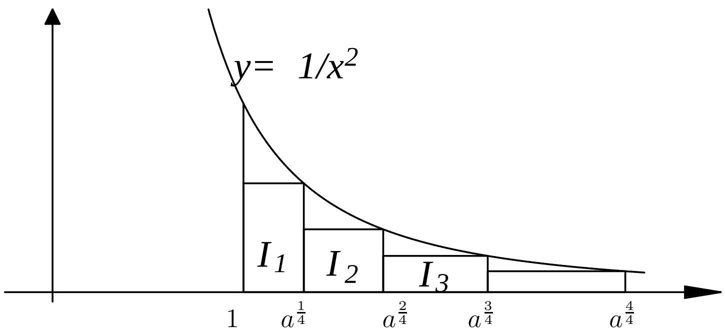
$$S_a \subset \bigcup_{j=1}^{n} O_j \tag{2.32}$$

so

$$\text{area}(S_a) \le \text{area}(\bigcup_{j=1}^{n} O_j) = b\left(1 - \frac{1}{b^n}\right). \tag{2.33}$$

Let I_j be the box

$$I_j = B\left(a^{\frac{j-1}{n}}, a^{\frac{j}{n}} : 0, \frac{1}{a^{\frac{2j}{n}}}\right) = B\left(b^{j-1}, b^j : 0, \frac{1}{b^{2j}}\right)$$



so that the upper right corner of I_j lies on the curve $y = \frac{1}{x^2}$ and I_j lies underneath the curve $y = \frac{1}{x^2}$. Then

$$\begin{aligned}\text{area}(I_j) &= \left(\frac{b^j - b^{j-1}}{b^{2j}}\right) = \frac{(b-1)\,b^{j-1}}{b^{2j}} \\ &= \frac{(b-1)}{b^{(j+1)}} = \frac{(b-1)}{b^2 b^{j-1}} = \frac{\text{area}(O_j)}{b^2}.\end{aligned}$$

Hence,

$$\begin{aligned}\text{area}\left(\bigcup_{j=1}^{n} I_j\right) &= \text{area}(I_1) + \cdots + \text{area}(I_n) \\ &= \frac{\text{area}(O_1)}{b^2} + \cdots + \frac{\text{area}(O_n)}{b^2} = \frac{(\text{area}(O_1) + \cdots + \text{area}(O_n))}{b^2} \\ &= \frac{1}{b^2}\text{area}\left(\bigcup_{j=1}^{n} O_j\right) = \frac{1}{b^2} \cdot b(1 - \frac{1}{b^n}) = b^{-1}(1 - b^{-n}).\end{aligned}$$

Since

$$\bigcup_{j=1}^{n} I_j \subset S_a,$$

we have

$$\text{area}\left(\bigcup_{j=1}^{n} I_j\right) \leq \text{area}(S_a);$$

i.e.,

$$b^{-1}(1 - b^{-n}) \leq \text{area}(S_a).$$

By combining this result with (2.33), we get

$$b^{-1}(1 - b^{-n}) \leq \text{area}(S_a) \leq b(1 - b^{-n}) \quad \text{for all} \quad n \in \mathbf{Z}^+.$$

Since $b = a^{\frac{1}{n}}$, we can rewrite this as

$$a^{-\frac{1}{n}}(1 - a^{-1}) \leq \text{area}(S_a) \leq a^{\frac{1}{n}}(1 - a^{-1}). \tag{2.34}$$

2.35 Exercise. What do you think the area of S_a should be? Explain your answer. If you have no ideas, take $a = 2$ in (2.34), take large values of n, and by using a calculator, estimate area(S_a) to three or four decimal places of accuracy.

2.36 Exercise. A Let a be a real number with $0 < a < 1$, and let N be a positive integer. Then

$$a = a^{\frac{N}{N}} < a^{\frac{N-1}{N}} < \cdots < a^{\frac{2}{N}} < a^{\frac{1}{N}} < 1.$$

Let T_a be the set of points (x, y) such that $a \leq x \leq 1$ and $0 \leq y \leq \dfrac{1}{x^2}$. Draw a sketch of T_a, and show that

$$a^{\frac{1}{N}}(a^{-1} - 1) \leq \text{area}(T_a) \leq a^{-\frac{1}{N}}(a^{-1} - 1).$$

The calculation of area(T_a) is very similar to the calculation of area(S_a).

What do you think the area of T_a should be?

2.37 Exercise. Using the inequalities (2.6), and the results of Bernoulli's table in section 2.2, try to guess what the area of S_a^r is for an arbitrary positive integer r. Explain the basis for your guess. (The correct formula for area(S_a^r) for positive integers r was stated by Bonaventura Cavalieri in 1647[6, 122 ff]. Cavalieri also found a method for computing general positive integer power sums.)

2.6 *Area of a Snowflake.

In this section we will find the areas of two rather complicated sets, called the *inner snowflake* and the *outer snowflake*. To construct the inner snowflake, we first construct a family of polygons I_1, I_2, $I_3 \dots$ as follows:

I_1 is an equilateral triangle.

I_2 is obtained from I_1 by adding an equilateral triangle to the middle third of each side of I_1, (see the snowflake figures 2.6).

I_3 is obtained from I_2 by adding an equilateral triangle to the middle third of each side of I_2, and in general

I_{n+1} is obtained from I_n by adding an equilateral triangle to the middle third of each side of I_n.

The inner snowflake is the set

$$K_I = \bigcup_{n=1}^{\infty} I_n,$$

i.e. a point is in the inner snowflake if and only if it lies in I_n for some positive integer n. Observe that the inner snowflake is not a polygon.

To construct the outer snowflake, we first construct a family of polygons O_1, O_2, $O_3 \dots$ as follows:

O_1 is a regular hexagon.

O_2 is obtained from O_1 by removing an equilateral triangle from the middle third of each side of O_1, (see the snowflake figures 2.6).

O_3 is obtained from O_2 by removing an equilateral triangle from the middle third of each side of O_2, and in general

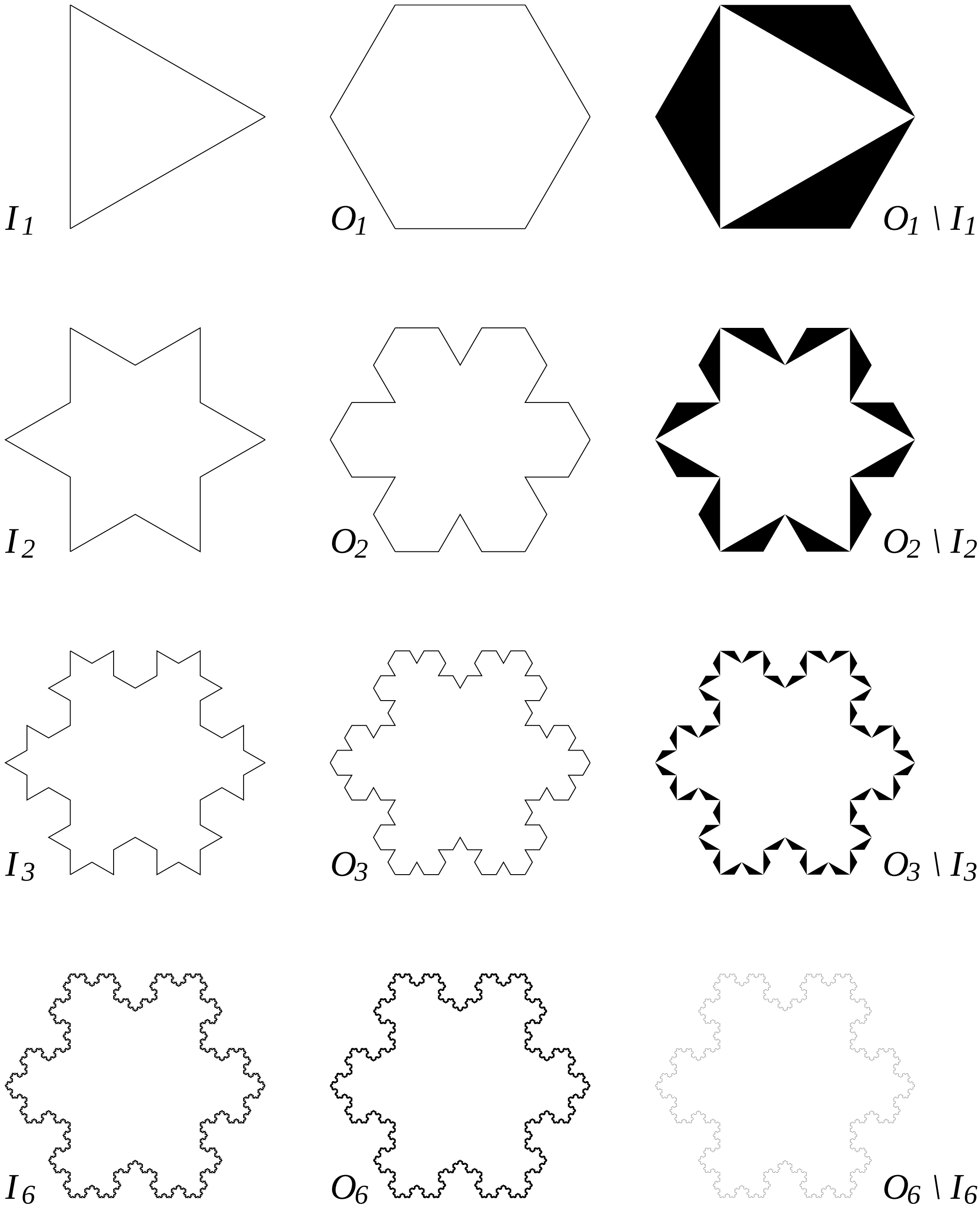
O_{n+1} is obtained from O_n by removing an equilateral triangle from the middle third of each side of O_n.

The outer snowflake is the set

$$K_O = \bigcap_{n=1}^{\infty} O_n,$$

i.e. a point is in the outer snowflake if and only if it lies in O_n for all positive integers n. Observe that the outer snowflake is not a polygon.

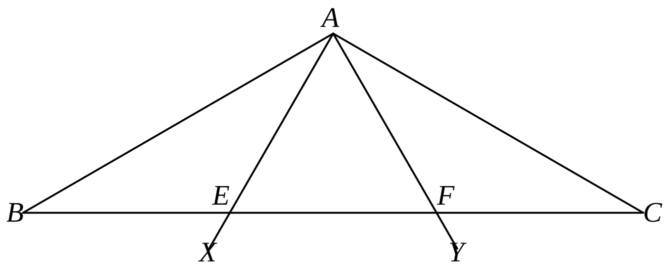
An isosceles 120° triangle is an isosceles triangle having a vertex angle of 120°. Since the sum of the angles of a triangle is two right angles, the base angles of such a triangle will be $\frac{1}{2}(180° - 120°) = 30°$.


I_1
O_1
$O_1 \setminus I_1$
I_2
O_2
$O_2 \setminus I_2$
I_3
O_3
$O_3 \setminus I_3$
I_6
O_6
$O_6 \setminus I_6$

Snowflakes

The following two technical lemmas[3] guarantee that in the process of building I_{n+1} from I_n we never reach a situation where two of the added triangles intersect each other, or where one of the added triangles intersects I_n, and in the process of building O_{n+1} from O_n we never reach a situation where two of the removed triangles intersect each other, or where one of the removed triangles fails to lie inside O_n.

2.38 Lemma. *Let $\triangle BAC$ be an isosceles $120°$ triangle with $\angle BAC = 120°$. Let E, F be the points that trisect BC, as shown in the figure. Then $\triangle AEF$ is an equilateral triangle, and the two triangles $\triangle AEB$ and $\triangle AFC$ are congruent isosceles $120°$ triangles.*



Proof: Let $\triangle BAC$ be an isosceles triangle with $\angle BAC = 120°$. Construct $30°$ angles BAX and CAY as shown in the figure, and let E and F denote the points where the lines AX and AY intersect BC. Then since the sum of the angles of a triangle is two right angles, we have

$$\angle AEB = 180° - \angle ABE - \angle BAE = 180° - 30° - 30° = 120°.$$

Hence

$$\angle AEF = 180° - \angle AEB = 180° - 120° = 60°,$$

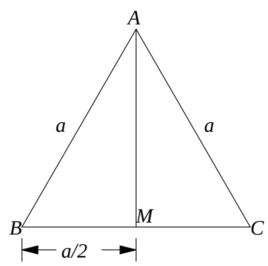
and similarly $\angle AFE = 60°$. Thus $\triangle AEF$ is an isosceles triangle with two $60°$ angles, and thus $\triangle AEF$ is equilateral. Now $\angle BAE = 30°$ by construction, and $\angle ABE = 30°$ since $\angle ABE$ is a base angle of an isosceles $120°$ triangle. It follows that $\triangle BEA$ is isosceles and $BE = EA$. (If a triangle has two equal angles, then the sides opposite those angles are equal.) Thus, $BE = EA = EF$, and a similar argument shows that $CF = EF$. It follows that the points E and F

[3] A lemma is a theorem which is proved in order to help prove some other theorem.

trisect BC, and that $\triangle AEB$ is an isosceles $120°$ triangle. A similar argument shows that $\triangle AFC$ is an isosceles $120°$ triangle.

Now suppose we begin with the isosceles $120°$ triangle $\triangle BAC$ with angle $BAC = 120°$, and we let E, F be the points that trisect BC. Since A and E determine a unique line, it follows from the previous discussion that EA makes a $30°$ angle with BA and FA makes a $30°$ angle with AC, and that all the conclusions stated in the lemma are valid. ‖

2.39 Lemma. *If T is an equilateral triangle with side of length a, then the altitude of T has length $\frac{a\sqrt{3}}{2}$, and the area of T is $\frac{\sqrt{3}}{4}a^2$. If R is an isosceles $120°$ triangle with two sides of length a, then the third side of R has length $a\sqrt{3}$.*



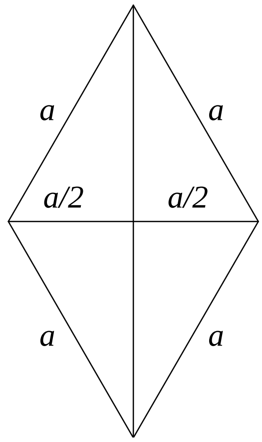
Proof: Let $T = \triangle ABC$ be an equilateral triangle with side of length a, and let M be the midpoint of BC. Then the altitude of T is AM, and by the Pythagorean theorem

$$AM = \sqrt{(AB)^2 - (BM)^2} = \sqrt{a^2 - \left(\frac{1}{2}a\right)^2} = \sqrt{\frac{3}{4}a^2} = \frac{\sqrt{3}}{2}a.$$

Hence

$$\text{area}(T) = \frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2}a \cdot \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{4}a^2.$$

An isosceles $120°$ triangle with two sides of length a can be constructed by taking halves of two equilateral triangles of side a, and joining them along their common side of length $\frac{a}{2}$, as indicated in the following figure.



Hence the third side of an isosceles 120° triangle with two sides of length a is twice the altitude of an equilateral triangle of side a, i.e., is $2\left(\frac{\sqrt{3}}{2}a\right) = \sqrt{3}a$. ‖

We now construct two sequences of polygons. $I_1, I_2, I_3, \cdots$, and $O_1, O_2, O_3, \cdots$ such that

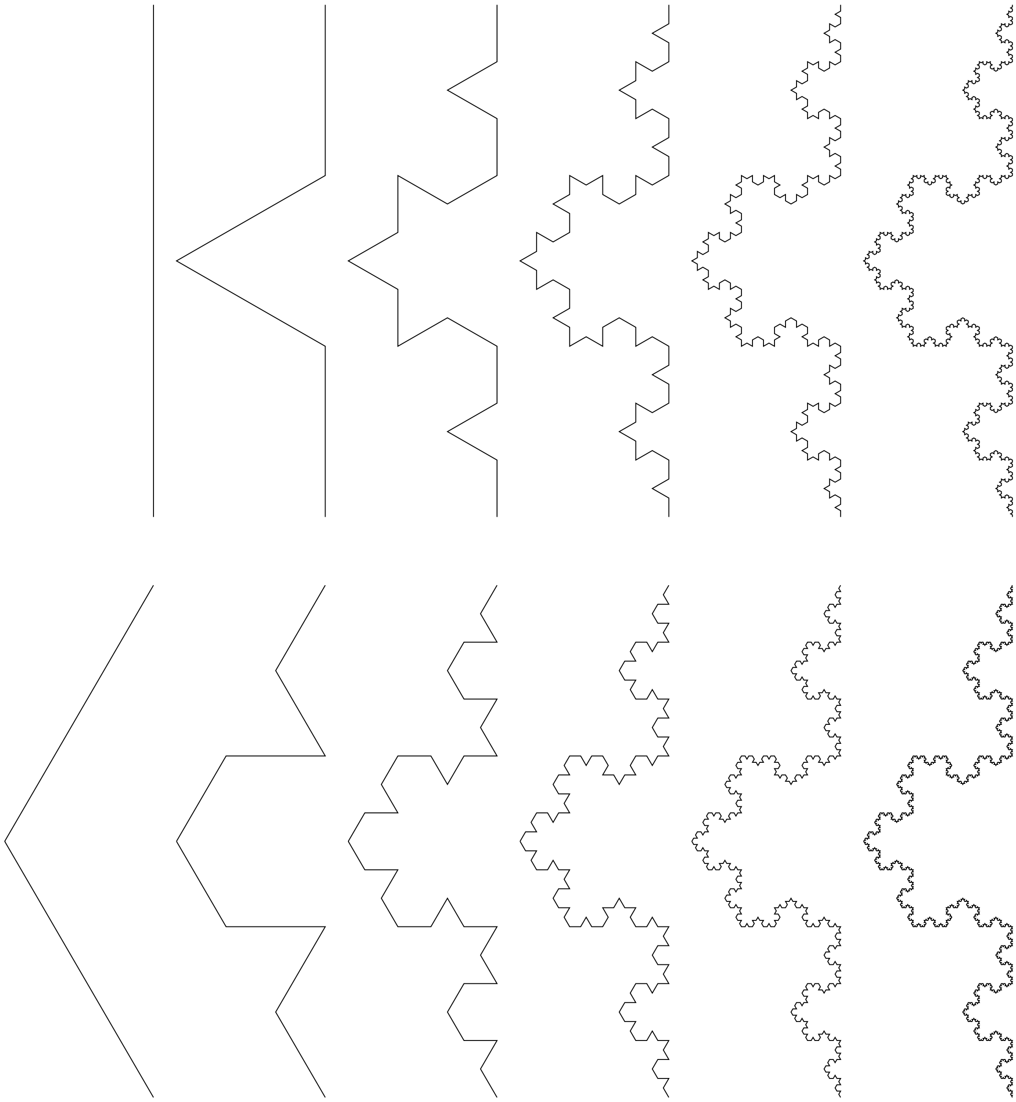
$$I_1 \subset I_2 \subset I_3 \subset \cdots \subset O_3 \subset O_2 \subset O_1.$$

Let O_1 be a regular hexagon with side 1, and let I_1 be an equilateral triangle inscribed in O_1. Then $O_1 \setminus I_1$ consists of three isosceles 120° triangles with short side 1, and from lemma 2.39, it follows that the sides of I_1 have length $\sqrt{3}$. (See the snowflake pictures 2.6.)

Our general procedure for constructing polygons will be:

subtriangle of $O_n \setminus I_n$ → subtriangles of $O_{n+1} \setminus I_{n+1}$

O_{n+1} is constructed from O_n by removing an equilateral triangle from the middle third of each side of O_n, and I_{n+1} is constructed from I_n by adding an equilateral triangle to the middle third of each side of I_n. For each n, $O_n \setminus I_n$ will consist of a family of congruent isosceles 120° triangles and $O_{n+1} \setminus I_{n+1}$ is obtained from $O_n \setminus I_n$ by removing an equilateral triangle from the middle third of each side of each isosceles 120° triangle. Pictures of I_n, O_n, and $O_n \setminus I_n$ are given in the figure 2.6. Details of the pictures are shown below.



Details of snowflakes

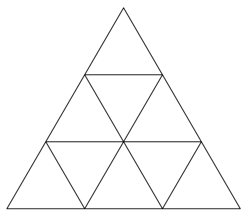
Lemma 2.38 guarantees that this process always leads from a set of isosceles 120° triangles to a new set of isosceles 120° triangles. Note that every vertex of O_n is a vertex of O_{n+1} and of I_{n+1}, and every vertex of I_n is a vertex of O_n and of I_{n+1}.
Let

$$\begin{aligned} s_n &= \text{length of a side of } I_n. \\ t_n &= \text{area of equilateral triangle with side } s_n. \\ m_n &= \text{number of sides of } I_n. \\ a_n &= \text{area of } I_n. \\ S_n &= \text{length of a side of } O_n. \\ T_n &= \text{area of equilateral triangle with side } S_n. \\ M_n &= \text{number of sides of } O_n. \\ A_n &= \text{area of } O_n. \end{aligned}$$

Then

$$\begin{array}{ll} s_{n+1} = \frac{1}{3}s_n, & S_{n+1} = \frac{1}{3}S_n, \\ m_{n+1} = 4m_n, & M_{n+1} = 4M_n, \\ a_{n+1} = a_n + m_n t_{n+1} & A_{n+1} = A_n - M_n T_{n+1}. \end{array}$$

Since an equilateral triangle with side s can be decomposed into nine equilateral triangles of side $\frac{s}{3}$ (see the figure),



we have

$$t_{n+1} = \frac{t_n}{9} \text{ and } T_{n+1} = \frac{T_n}{9}.$$

Also

$$a_1 = \text{area}(I_1) = t_1,$$

and since O_1 can be written as a union of six equilateral triangles,

$$A_1 = 6T_1.$$

The following table summarizes the values of s_n, m_n, t_n, S_n, M_n and T_n:

n	m_n	t_n	$m_{n-1}t_n$	M_n	T_n	$M_{n-1}T_n$
1	3	a_1		6	$\frac{A_1}{6}$	
2	$3 \cdot 4$	$\frac{a_1}{9}$	$\frac{3}{9}a_1$	$6 \cdot 4$	$\frac{1}{9}\frac{A_1}{6}$	$\frac{A_1}{9}$
3	$3 \cdot 4^2$	$\frac{a_1}{9^2}$	$\frac{3}{9} \cdot \frac{4}{9}a_1$	$6 \cdot 4^2$	$\frac{1}{9^2}\frac{A_1}{6}$	$\frac{4}{9}\frac{A_1}{9}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$3 \cdot 4^{n-1}$	$\frac{a_1}{9^{n-1}}$	$\frac{3}{9}\left(\frac{4}{9}\right)^{n-2} a_1$	$6 \cdot 4^{n-1}$	$\frac{1}{9^{n-1}}\frac{A_1}{6}$	$\left(\frac{4}{9}\right)^{n-2}\frac{A_1}{9}$

Now

$$\begin{aligned}
A_2 &= A_1 - M_1T_2 = A_1 - \frac{A_1}{9}, \\
A_3 &= A_2 - M_2T_3 = A_1 - \frac{A_1}{9} - \left(\frac{4}{9}\right)\frac{A_1}{9}, \\
&\vdots \\
A_{n+1} &= A_n - M_nT_{n+1} \\
&= A_1 - \frac{A_1}{9} - \left(\frac{4}{9}\right)\frac{A_1}{9} - \left(\frac{4}{9}\right)^2\frac{A_1}{9} - \cdots - \left(\frac{4}{9}\right)^{n-1}\frac{A_1}{9} \\
&= A_1 - \frac{A_1}{9}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right). \qquad (2.40)
\end{aligned}$$

Also,

$$\begin{aligned}
a_2 &= a_1 + m_1t_2 = a_1 + \frac{3}{9}a_1, \\
a_3 &= a_2 + m_2t_3 = a_1 + \frac{3}{9}a_1 + \frac{3}{9}\left(\frac{4}{9}\right)a_1, \\
&\vdots \\
a_{n+1} &= a_n + m_nt_{n+1} \\
&= a_1 + \frac{3}{9}a_1 + \frac{3}{9}\left(\frac{4}{9}\right)a_1 + \frac{3}{9}\left(\frac{4}{9}\right)^2 a_1 + \cdots + \frac{3}{9}\left(\frac{4}{9}\right)^{n-1} a_1 \\
&= a_1 + \frac{a_1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right). \qquad (2.41)
\end{aligned}$$

By the formula for a finite geometric series we have

$$1+\frac{4}{9}+(\frac{4}{9})+\cdots+(\frac{4}{9})^{n-1}=\frac{1-(\frac{4}{9})^n}{1-\frac{4}{9}}=\frac{9}{5}\left[1-(\frac{4}{9})^n\right].$$

By using this result in equations (2.40) and (2.41) we obtain

$$\begin{aligned}\text{area}(O_{n+1}) &= A_{n+1}=A_1-\frac{A_1}{5}\left[1-\left(\frac{4}{9}\right)^n\right]\\ &= \frac{4}{5}A_1+\frac{A_1}{5}\left(\frac{4}{9}\right)^n, \qquad (2.42)\end{aligned}$$

and

$$\begin{aligned}\text{area}(I_{n+1}) &= a_{n+1}=a_1+\frac{a_1}{3}\cdot\frac{9}{5}\left[1-\left(\frac{4}{9}\right)^n\right]\\ &= \frac{8}{5}a_1-\frac{3a_1}{5}\left(\frac{4}{9}\right)^n.\end{aligned}$$

Now you can show that $a_1=\dfrac{A_1}{2}$, so the last equation may be written as

$$\text{area}(I_{n+1})=\frac{4}{5}A_1-\frac{3a_1}{5}\left(\frac{4}{9}\right)^n. \qquad (2.43)$$

2.44 Exercise. Show that $a_1=\dfrac{A_1}{2}$, i.e. show that $\text{area}(I_1)=\dfrac{1}{2}\text{area}(O_1)$.

2.45 Definition (Snowflakes.) Let $K_I=\bigcup_{n=1}^{\infty}I_n$ and $K_O=\bigcap_{n=1}^{\infty}O_n$. Here the *infinite union* $\bigcup_{n=1}^{\infty}I_n$ means the set of all points x such that $x\in I_n$ for some n in $\mathbf{Z}^+$, and the *infinite intersection* $\bigcap_{n=1}^{\infty}O_n$ means the set of points x that are in all of the sets O_n where $n\in\mathbf{Z}^+$. I will call the sets K_I and K_O the *inner snowflake* and the *outer snowflake*, respectively.

For all k in $\mathbf{Z}^+$, we have

$$I_k\subset\bigcup_{n=1}^{\infty}I_n=K_I\subset K_O=\bigcap_{n=1}^{\infty}O_n\subset O_k,$$

so

$$\text{area}(I_k) \leq \text{area}(K_I) \leq \text{area}(K_O) \leq \text{area}(O_k).$$

Since $\left(\dfrac{4}{9}\right)^n$ can be made very small by taking n large (see theorem 6.66), we conclude from equations 2.43 and 2.42 that

$$\text{area}(K_I) = \text{area}(K_O) = \frac{4}{5}A_1 = \frac{4}{5}\text{area}(O_1).$$

We will call O_1 the *circumscribed hexagon* for K_I and for K_O. We have proved the following theorem:

2.46 Theorem. *The area of the inner snowflake and the outer snowflake are both* $\frac{4}{5}$ *of the area of the circumscribed hexagon.*

Note that both snowflakes touch the boundary of the circumscribed hexagon in infinitely many points.

It is natural to ask whether the sets K_O and K_I are the same.

2.47 Entertainment (Snowflake Problem.) Show that the inner snowflake is not equal to the outer snowflake. In fact, there are points in the boundary of the circumscribed hexagon that are in the outer snowflake but not in the inner snowflake.

The snowflakes were discovered by Helge von Koch(1870–1924), who published his results in 1906 [31]. Actually Koch was not interested in the snowflakes as two-dimensional objects, but as one-dimensional curves. He considered only part of the boundary of the regions we have described. He showed that the boundary of K_O and K_I is a curve that does not have a tangent at any point. You should think about the question: "In what sense is the boundary of K_O a curve?" In order to answer this question you would need to answer the questions "what is a curve?" and "what is the boundary of a set in $\mathbf{R}^2$?" We will not consider these questions in this course, but you might want to think about them.

I will leave the problem of calculating the perimeter of a snowflake as an exercise. It is considerably easier than finding the area.

2.48 Exercise. Let I_n and O_n be the polygons described in section 2.6, which are contained inside and outside of the snowflakes K_I and K_O.

a) Calculate the length of the perimeter of I_n.

b) Calculate the length of the perimeter of O_n.

What do you think the perimeter of K_O should be? (Since it isn't really clear what we mean by "the perimeter of K_O," this question doesn't really have a "correct" answer – but you should come up with some answer.)