Problem 1 | | Problem set 9

For $\psi(r) = \frac{Ae^{ikr}}{r}$, we have $\nabla^2\psi = \frac{1}{r}(r\psi)'' = \frac{1}{r}(-k^2 A e^{ikr}) = -k^2\psi$

& the time-independent Schrödinger eqn. is:

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \Rightarrow E = \frac{k^2\hbar^2}{2m}, \text{ so } \Psi(\vec{r},t) = \psi(\vec{r})e^{-iEt/\hbar} = \frac{Ae^{ikr}}{r}\cdot e^{-i\frac{k^2\hbar t}{2m}}$$

or

$$\Psi(\vec{r},t) = \frac{A}{r}e^{ik(r - \frac{k\hbar t}{2m})}$$ & thus satisfies $-\frac{\hbar^2}{2m}\nabla^2\Psi = i\hbar\frac{\partial\Psi}{\partial t}$

Problem 2

a. We need to solve: $\left(\frac{d^2}{dx^2} + k^2\right)G(x) = 0 \Rightarrow G(x) = Ae^{ikx} + Be^{-ikx}$

we'll take: $G_+(x) = Ae^{ikx}$, the outgoing solution for $x>0$, $G_-(x) = A e^{-ikx}$ for $x<0$.

b. Integrating: $\int_{-\epsilon}^{\epsilon}\frac{d^2G}{dx^2}dx + k^2\int_{-\epsilon}^{\epsilon}G(x)dx = 1$

$$G'(\epsilon) - G'(-\epsilon) + k^2\int_{-\epsilon}^{0}G_-(x)dx + k^2\int_0^{\epsilon}G_+(x)dx = 1$$

or

$$G_+'(\epsilon) - G_-'(-\epsilon) + k^2\left[-\frac{1}{ik}(G_-(0) - G_-(\epsilon)) + \frac{1}{ik}(G_+(\epsilon) - G_-(0))\right] = 1$$

$$ikG_+(\epsilon) + ikG_-(-\epsilon) + ik[G_-(0) - G_-(\epsilon) - G_+(\epsilon) + G_+(0)] = 1$$

so we have: $2ikA = 1 \Rightarrow A = \frac{1}{2ik}$.

we can combine the G_+ & G_- solutions: $G(x) = \frac{e^{ik|x|}}{2ik}$

then $G(x,x') = \frac{e^{ik|x-x'|}}{2ik}$.

Problem 3

The integral form of the solution is:

$$P(x) = \int_{-\infty}^{+\infty}G(x,x')\cdot V(x')dx' = \int_{-\infty}^{+\infty}\frac{e^{ik|x-x'|}}{2ik}V(x')dx'$$

For $x'<x$, $|x-x'| = x-x'$ & for $x'>x$, $|x-x'| = x'-x$

&

$$P(x) = \int_{-\infty}^{x}\frac{e^{ik(x-x')}}{2ik}V(x')dx' + \int_x^{\infty}\frac{e^{-ik(x-x')}}{2ik}V(x')dx'$$

$$= \frac{e^{ikx}}{2ik}\int_{-a}^{x}V_0e^{-ikx'}dx' + \frac{e^{-ikx}}{2ik}\int_x^{a}V_0e^{ikx'}dx'$$

Problem 3 (continued)

carrying out the integration:

$$P(x) = \frac{e^{ikx}}{2ik}\left(\frac{-V_0}{ik}\right)\left[e^{-ikx} - e^{ika}\right] + \frac{e^{-ikx}}{2ik}\left(\frac{V_0}{ik}\right)\left[e^{ika} - e^{ikx}\right]$$

$$= \frac{V_0}{2k^2}\left(1 - e^{ik(a+x)}\right) - \frac{V_0}{2k^2}\left(e^{ik(a-x)} - 1\right)$$

$$= \frac{V_0}{2k^2}\left(2 - e^{ika}\left(e^{ikx} + e^{-ikx}\right)\right) = \frac{V_0}{k^2}\left(1 - e^{ika}\cos(kx)\right)$$

& this has: $P''(x) + k^2 P(x) = V_0$ ✓

Problem 4

Using $f(\theta) = -\frac{2m}{\hbar^2 K}\int_0^\infty r V(r)\sin(Kr)\,dr$ w/ $K \equiv 2k\sin(\theta/2)$ & $k^2 \equiv \frac{2mE}{\hbar^2}$

w/ $V(r) = V_0\,\Theta(R-r)$, we have

$$f(\theta) = -\frac{2m}{\hbar^2 K}\int_0^R V_0\, r\sin(Kr)\,dr$$

Note that $\frac{d}{dr}\left(Ar\cos(Kr) + B\sin(Kr)\right) = A\cos(Kr) - ArK\sin(Kr) + BK\cos(Kr)$

$$= (A+BK)\cos(Kr) - AKr\sin(Kr)$$

so let $A = -\frac{V_0}{K}$ & $B = -A/K = +V_0/K^2$, then

$\frac{d}{dr}\left(-\frac{V_0}{K}r\cos(Kr) + \frac{V_0}{K^2}\sin(Kr)\right) = V_0 r\sin(Kr)$, the integrand, so

$$f(\theta) = -\frac{2m}{\hbar^2 K}\left[\int_0^R V_0 r\sin(Kr)\,dr\right] = \frac{2mV_0}{\hbar^2 K^2}\left[R\cos(KR) - \frac{1}{K}\sin(KR)\right]$$

(I used R instead of "a") for the sphere's radius)

"low energy" means: $|\vec{K}\cdot\vec{r}_0| \ll 1 \Rightarrow KR \ll 1$, small angle approx'n

$$f(\theta) \approx \frac{2mV_0}{\hbar^2 K^2}\Big[\underbrace{R(1-\tfrac{1}{2}K^2R^2) - \tfrac{1}{K}(KR - \tfrac{1}{6}(KR)^3)}_{= -\frac{2}{6}K^2R^3}\Big] = -\frac{2mV_0R^3}{3\hbar^2}$$

Problem 5

We have: $f(\theta) = -\frac{2m\beta}{\hbar^2 k}\int_0^\infty e^{-\mu r}\sin(kr)\,dr$

& the integral can be written in terms of exponentials:

$$\int_0^\infty e^{-\mu r}\sin(kr)\,dr = \frac{1}{2i}\int_0^\infty \left[e^{-\mu r + ikr} - e^{-\mu r - ikr}\right]dr = \frac{1}{2i}\left[\frac{e^{-\mu r}e^{ikr}}{-\mu+ik}\Big|_0^\infty + \frac{e^{-\mu r}\cdot e^{-ikr}}{\mu+ik}\Big|_0^\infty\right]$$

$$= \frac{-1}{2i}\left[\frac{1}{-\mu+ik} + \frac{1}{\mu+ik}\right] = \frac{-1}{2i}\left(\frac{\mu+ik-\mu+ik}{-\mu^2-k^2}\right) = +\frac{k}{\mu^2+k^2}$$

so $f(\theta) = \frac{-2m\beta}{\hbar^2(\mu^2+k^2)}$

Problem 6

The differential cross section is: $D(\theta) = |f(\theta)|^2 = \frac{4m^2\beta^2}{\hbar^4(\mu^2+4k^2\sin^2(\theta/2))^2}$

so

$$\sigma = \int_0^{2\pi}\int_0^{\pi} |f(\theta)|^2 \sin\theta\,d\theta\,d\phi = \frac{2\pi\cdot 4m^2\beta^2}{\mu^4\hbar^4}\int_0^\pi \frac{\sin\theta}{\left(1+\underbrace{\left(\frac{2k}{\mu}\right)^2}_{\equiv\alpha}\sin^2(\theta/2)\right)^2}d\theta$$

& $\int \frac{\sin\theta}{(1+\alpha\sin^2\theta/2)^2}d\theta = \frac{-4}{\alpha(2+\alpha-\alpha\cos\theta)}$ so $\int_0^\pi \frac{\sin\theta}{(1+\alpha\sin^2\theta/2)^2}d\theta = -\frac{4}{\alpha(2+2\alpha)} + \frac{4}{2\alpha} = \frac{2}{1+\alpha}$

then (using $k^2 = 2mE/\hbar^2$

$$\sigma = \frac{8\pi m^2\beta^2}{\mu^4\hbar^4}\cdot\frac{2}{\left(1+\frac{4k^2}{\mu^2}\right)} = \frac{16\pi m^2\beta^2}{\mu^2\hbar^2(\mu^2\hbar^2+8mE)}$$

Problem 7

For $f(x,x') = \sum_{j=1}^{\infty} \psi_j^*(x)\psi_j(x')$, we'll show that: $\int_{-\infty}^{+\infty} h(x) f(x,x') dx = h(x')$

to establish that $f(x,x')$ behaves like $\delta(x-x')$.

Decompose $h(x)$ in the $\{\psi_j(x)\}_{j=1}^{\infty}$ basis: $h(x) = \sum_{k=1}^{\infty} a_k \psi_k(x)$.

Then
$$\begin{aligned}\int_{-\infty}^{+\infty} f(x,x')h(x)dx &= \sum_{k=1}^{\infty} a_k \int_{-\infty}^{+\infty} \sum_{j=1}^{\infty} \psi_j^*(x)\psi_j(x')\psi_k(x)\,dx \\ &= \sum_{k=1}^{\infty} a_k \sum_{j=1}^{\infty} \psi_j(x') \underbrace{\int_{-\infty}^{+\infty} \psi_j^*(x)\psi_k(x)dx}_{=\delta_{jk}} \\ &= \sum_{k=1}^{\infty} a_k \psi_k(x') = h(x') \checkmark\end{aligned}$$