Problem 1 Problem Set 8

For $\psi = Ae^{-br^2}$, normalize:

$$\int_0^\infty \int_0^{2\pi} \int_0^\pi A^2 e^{-2br^2} \cdot r^2 \sin\theta \, d\theta \, d\phi \, dr = 4\pi A^2 \overbrace{\int_0^\infty e^{-2br^2} r^2 dr}^{\text{back cover of book}} = A^2 \left(\frac{\pi}{2b}\right)^{3/2} = 1$$

so $A = \left(\frac{2b}{\pi}\right)^{3/4}$. The $\hat{H}$ operator is: $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r}$

w/ $\nabla^2 \psi = \frac{2}{r}\psi' + \psi''$

Then $\langle \psi | \nabla^2 | \psi \rangle = \int_0^\infty \int_0^{2\pi} \int_0^\pi \left(\frac{2b}{\pi}\right)^{3/2} \cdot e^{-br^2} \left[\frac{2}{r}\left(-2bre^{-br^2}\right) + \left(-2be^{-br^2} - 2br\left(-2bre^{-br^2}\right)\right)\right] r^2 \sin\theta \, d\theta \, d\phi \, dr$

$$= 4\pi \left(\frac{2b}{\pi}\right)^{3/2} \int_0^\infty e^{-br^2}\left[-6be^{-br^2} + 4b^2r^2e^{-br^2}\right] \cdot r^2 dr$$

$$= 4\pi \left(\frac{2b}{\pi}\right)^{3/2} \int_0^\infty \left[-6br^2 + 4b^2r^4\right] e^{-2br^2} dr \quad \leftarrow \text{back cover}$$

$$= -4\pi \left(\frac{2b}{\pi}\right)^{3/2} \cdot \frac{3}{8} \left(\frac{\pi}{2b}\right)^{1/2} = -3b$$

& $\langle \psi | \hat{T} | \psi \rangle = \frac{3b\hbar^2}{2m}$

$\hat{T} \equiv \frac{\hat{p}^2}{2m}$.

& (back cover)

$$\langle \psi | \frac{1}{r} | \psi \rangle = 4\pi \left(\frac{2b}{\pi}\right)^{3/2} \int_0^\infty \frac{1}{r} e^{-2br^2} \cdot r^2 dr = 4\pi \left(\frac{2b}{\pi}\right)^{3/2} \cdot \frac{1}{4b} = 2\sqrt{\frac{2b}{\pi}}$$

so $\langle \psi | U | \psi \rangle = -\frac{e^2}{4\pi\epsilon_0} \cdot 2\sqrt{\frac{2b}{\pi}} = -\frac{e^2\sqrt{\frac{b}{2}}}{\epsilon_0 \pi^{3/2}}$

$U = \frac{-e^2}{4\pi\epsilon_0 r}$

and the expectation value is: $\langle \psi | \hat{H} | \psi \rangle \equiv E_b = \frac{3b\hbar^2}{2m} - \frac{e^2\sqrt{b}}{\sqrt{2}\,\epsilon_0 \pi^{3/2}}$

Now we minimize: $\frac{dE_b}{db} = \frac{3\hbar^2}{2m} - \frac{\frac{1}{2}e^2}{\sqrt{2}\,\epsilon_0\pi^{3/2}} \frac{1}{\sqrt{b}} = 0$

or, solving for b: $\sqrt{b} = \frac{\frac{1}{2}e^2 \cdot 2m}{3\hbar^2\sqrt{2}\,\epsilon_0\pi^{3/2}} \Rightarrow b = \frac{m^2e^4}{18\hbar^4\epsilon_0^2\pi^3}$

w/ $E_{b\,min} = -\frac{me^4}{12\pi^3\epsilon_0^2\hbar^2} \equiv E_V$. The hydrogen ground state is $E_1 = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2}$

so

$E_V = \frac{8}{3\pi}E_1$ & $\frac{8}{3\pi} \approx .85$ so $E_V = -11.6\,eV$ (vs. $-13.6\,eV$ for hydrogen)

Problem 2 (8.4b)

back cover

For $\psi(x) = A x e^{-bx^2}$, we normalize: $\int_{-\infty}^{+\infty} A^2 x^2 e^{-2bx^2} dx = A^2 \cdot \frac{1}{4}\sqrt{\frac{\pi}{2b^3}} = 1 \Rightarrow A^2 = 4\left(\frac{2b^3}{\pi}\right)^{1/2}$

The Hamiltonian is $\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$

For the kinetic piece: $\frac{d\psi}{dx} = Ae^{-bx^2} - 2Abx^2 e^{-bx^2}$

$$\frac{d^2\psi}{dx^2} = -2bxAe^{-bx^2} - 4Abxe^{-bx^2} + 4Ab^2x^3e^{-bx^2}$$

then

$$\langle\psi|\hat{T}|\psi\rangle = -\frac{A^2\hbar^2}{2m}\left[\int_{-\infty}^{+\infty}\left(-6bx^2e^{-2bx^2} + 4b^2x^4e^{-bx^2}\right)dx\right]$$ (back cover)

$$= -\frac{A^2\hbar^2}{2m}\left[-6b\cdot 2\cdot\sqrt{\pi}\cdot 2\left(\frac{1}{2\sqrt{2b}}\right)^3 + 4b^2\cdot 2\sqrt{\pi}\,\frac{4!}{2!}\left(\frac{1}{2\sqrt{2b}}\right)^5\right]$$

$$= -\frac{\hbar^2}{2m}\cdot 4(2b^3)^{1/2}\left[-\frac{3b}{(2b)^{3/2}} + \frac{3b^2}{(2b)^{5/2}}\right] = -\frac{2\hbar^2}{m}\sqrt{2}\,b\left[-\frac{3}{2^{3/2}} + \frac{3}{2^{5/2}}\right]$$

$$= -\frac{\hbar^2}{m}\cdot 3b\left[-1 + \frac{1}{2}\right] = \frac{3b\hbar^2}{2m}$$

The potential piece is: $\langle\psi|U|\psi\rangle = A^2\int_{-\infty}^{+\infty}\frac{1}{2}m\omega^2 x^4 e^{-2bx^2}dx$

$$= \frac{1}{2}m\omega^2A^2\left[\sqrt{\pi}\cdot\frac{4!}{2!}\cdot 2\left(\frac{1}{2\sqrt{2b}}\right)^5\right] = \frac{2m\omega^2}{\sqrt{\pi}}\sqrt{2b^3}\left[\sqrt{\pi}\,\frac{3}{4}\,\frac{1}{(2b)^{5/2}}\right]$$

$$= \frac{3m\omega^2}{2}\cdot\frac{\sqrt{2}\,b^{3/2}}{2^{5/2}\,b^{5/2}} = \frac{3m\omega^2}{8b}$$

and

$E_b = \langle\psi|\hat{H}|\psi\rangle = \frac{3b\hbar^2}{2m} + \frac{3m\omega^2}{8b}$, now minimizing using b:

$$\frac{dE_b}{db} = \frac{3\hbar^2}{2m} - \frac{3m\omega^2}{8b^2} = 0 \Rightarrow b^2 = \frac{6m^2\omega^2}{24\hbar^2} \Rightarrow b = \frac{m\omega}{2\hbar}$$

the minimum is $E_{b\,min} = \frac{3\hbar^2}{2m}\left(\frac{m\omega}{2\hbar}\right) + \frac{3m\omega^2}{8}\cdot\frac{2\hbar}{m\omega} = \frac{3}{4}\hbar\omega + \frac{3}{4}\hbar\omega = \frac{3}{2}\hbar\omega$ ✓

Problem 3 (8.7)

For (++) ⊖ we know $E = 4E_1$ (w/ $E_1 \cong -13.6\,\text{eV}$), so for a single electron, $E = 4E_1 \cong -54.4\,\text{eV}$, the difference between the ground state & this gives the ionization energy:

$$-54.4\,\text{eV} - (-79\,\text{eV}) \cong 24.6\,\text{eV}.$$

Problem 4

For the δ well, the single bound state is: $\psi_b = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}$

The scattering states are
$$\psi_s = \begin{cases} A\left(e^{ikx} + \frac{i\beta}{1-i\beta} e^{-ikx}\right) & x \leq 0 \\ A \cdot \frac{1}{1-i\beta} e^{ikx} & x \geq 0. \end{cases} \qquad \beta \equiv \frac{m\alpha}{\hbar^2 k},\ k > 0$$

we'll verify that $\langle \psi_b | \psi_s \rangle = 0$,

$$\langle \psi_b | \psi_s \rangle = \int_{-\infty}^{+\infty} \psi_b^*(x)\, \psi_s(x)\, dx$$

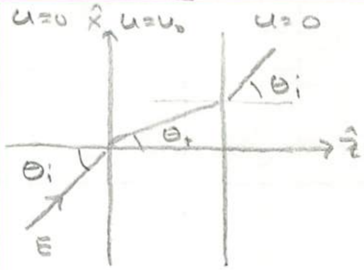
$$= \frac{A\sqrt{m\alpha}}{\hbar}\left[\int_{-\infty}^{0} e^{x(ik+k\beta)} dx + \frac{i\beta}{1-i\beta}\int_{-\infty}^{0} e^{x(-ik+k\beta)} dx + \frac{1}{1-i\beta}\int_{0}^{\infty} e^{x(ik-k\beta)} dx\right]$$

$$= \frac{A\sqrt{m\alpha}}{\hbar}\left[\frac{1}{ki(1-i\beta)} + \frac{i\beta}{(1-i\beta)} \cdot \frac{1}{(-ki(1+i\beta))} - \frac{1}{(1-i\beta)} \cdot \frac{1}{(ki(1+i\beta))}\right]$$

$$= \frac{A\sqrt{m\alpha}}{\hbar}\left[\underbrace{\frac{-i}{k(1-i\beta)}} - \frac{\beta}{k(1+\beta^2)} + \frac{i}{k(1+\beta^2)}\right] = 0$$

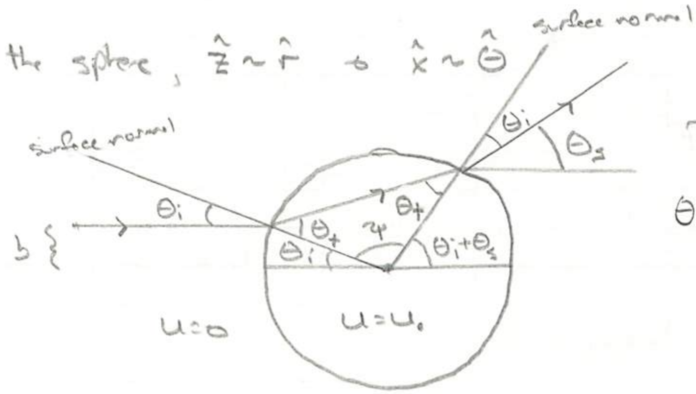
$$= \frac{-i(1+i\beta)}{k(1+\beta^2)} = \frac{-i}{k(1+\beta^2)} + \frac{\beta}{k(1+\beta^2)}$$

Problem 5



$$\sin\theta_t = \frac{\sin\theta_i}{\sqrt{1-u_0/E}}$$

For the sphere, $\hat{z} \sim \hat{r}$ & $\hat{x} \sim \hat{\theta}$



$$\psi + 2\theta_t = \pi \Rightarrow \psi = \pi - 2\theta_t$$

$$\theta_i + \psi + (\theta_i + \theta_s) = \pi \Rightarrow \theta_s = 2(\theta_t - \theta_i)$$

Now $\sin\theta_i = b/R$, & $\sin\theta_t = \dfrac{\sin\theta_i}{\sqrt{1-u_0/E}} = \underbrace{\dfrac{1}{\sqrt{1-u_0/E}}}_{\equiv \alpha} \dfrac{b}{R}$

& $\theta_s = 2(\theta_t - \theta_i) = 2\left(\sin^{-1}(\alpha b/R) - \sin^{-1}(b/R)\right)$