Starting w/ the state w/ $\hat{L}_z|\psi\rangle = \hbar(1/2+1/2)|\psi\rangle$, the only possibility is

$$|\psi\rangle = |1/2\ 1/2\ 1/2\ 1/2\rangle$$

To check the value of $\hat{L}^2$ acting on $|\psi\rangle$, note that

$$\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + 2\hat{L}_1\cdot\hat{L}_2 \quad \& \quad \hat{L}_1\cdot\hat{L}_2 = \hat{L}_{1x}\hat{L}_{2x} + \hat{L}_{1y}\hat{L}_{2y} + \hat{L}_{1z}\hat{L}_{2z}$$

& $\hat{L}_{1x} = 1/2(\hat{L}_{1+} + \hat{L}_{1-})$, $\hat{L}_{1y} = \frac{1}{2i}(\hat{L}_{1+} - \hat{L}_{1-})$ & sim. for $\hat{L}_{2x}$ & $\hat{L}_{2y}$

$$\begin{aligned}
\hat{L}_1\cdot\hat{L}_2|\psi\rangle &= 1/4(\hat{L}_{1+}+\hat{L}_{1-})(\hat{L}_{2+}+\hat{L}_{2-})|\psi\rangle - 1/4(\hat{L}_{1+}-\hat{L}_{1-})(\hat{L}_{2+}-\hat{L}_{2-})|\psi\rangle + \hat{L}_{1z}\hat{L}_{2z}|\psi\rangle \\
&= 1/4\,\hat{L}_{1-}\hat{L}_{2-}|\psi\rangle - 1/4\,\hat{L}_{1-}\hat{L}_{2-}|\psi\rangle + \hbar^2(1/2)^2|\psi\rangle \\
&= 1/4\,\hbar^2|\psi\rangle
\end{aligned}$$

So $\hat{L}^2|\psi\rangle = \left(\hbar^2\, 2\cdot 1/2(1/2+1) + 1/2\,\hbar^2\right)|\psi\rangle = 2\hbar^2|\psi\rangle \Rightarrow L=1$ ✓

The state $|1\,1\rangle = |1/2\ 1/2\ 1/2\ 1/2\rangle$.

Now acting w/ the lowering operator: $\hat{L}_- = \hat{L}_x - i\hat{L}_y = \hat{L}_{1x} - i\hat{L}_{1y} + \hat{L}_{2x} - i\hat{L}_{2y} = \hat{L}_{1-} + \hat{L}_{2-}$

$$\hat{L}_-|1\,1\rangle = (\hat{L}_{1-}+\hat{L}_{2-})|1/2\ 1/2\ 1/2\ 1/2\rangle = \hbar\sqrt{1/2(1/2+1) - 1/2(1/2-1)}\left(|1/2\ 1/2\ {-1/2}\ 1/2\rangle + |1/2\ 1/2\ 1/2\ {-1/2}\rangle\right)$$

$$\hbar\sqrt{1(1+1)-1(1-1)}\,|1\,0\rangle = \hbar\left(|1/2\ 1/2\ {-1/2}\ 1/2\rangle + |1/2\ 1/2\ 1/2\ {-1/2}\rangle\right)$$

giving $\quad |1\,0\rangle = \frac{1}{\sqrt{2}}\left(|1/2\ 1/2\ {-1/2}\ 1/2\rangle + |1/2\ 1/2\ 1/2\ {-1/2}\rangle\right)$

Using $\hat{L}_-$ again:

$$\hat{L}_-|1\,0\rangle = \frac{1}{\sqrt{2}}\hbar\sqrt{1/2(1/2+1) - 1/2(1/2-1)}\left(|\tfrac12\,\tfrac12\,{-\tfrac12}\,{-\tfrac12}\rangle + |\tfrac12\,\tfrac12\,{-\tfrac12}\,{-\tfrac12}\rangle\right)$$

$$\hbar\sqrt{2}\,|1\,{-1}\rangle = \hbar\sqrt{2}\,|\tfrac12\,\tfrac12\,{-\tfrac12}\,{-\tfrac12}\rangle \Rightarrow |1\,{-1}\rangle = |\tfrac12\,\tfrac12\,{-\tfrac12}\,{-\tfrac12}\rangle$$

Problem 2

a. From the table: $|1\,1\rangle = |\tfrac12\,\tfrac12\,\tfrac12\,\tfrac12\rangle$, $|1\,0\rangle = 1/\sqrt{2}\left(|\tfrac12\,\tfrac12\,\tfrac12\,{-\tfrac12}\rangle + |\tfrac12\,\tfrac12\,{-\tfrac12}\,\tfrac12\rangle\right)$
$|1\,{-1}\rangle = |\tfrac12\,\tfrac12\,{-\tfrac12}\,{-\tfrac12}\rangle$ ✓

b. $|0\,0\rangle = 1/\sqrt{2}\left(|\tfrac12\,\tfrac12\,\tfrac12\,{-\tfrac12}\rangle - |\tfrac12\,\tfrac12\,{-\tfrac12}\,\tfrac12\rangle\right)$

c. $|2\,{-1}\rangle = \sqrt{3/4}\,|3/2\ 1/2\ {-1/2}\ {-1/2}\rangle + \sqrt{1/4}\,|3/2\ 1/2\ {-3/2}\ 1/2\rangle$

Problem 4 (6.22) (omitted)

b. $\langle l'm'|[\hat{L}_z,\hat{V}_z]|lm\rangle = 0 = \underbrace{\langle l'm'|\hat{L}_z}_{=\hbar m'\langle l'm'|}\hat{V}_z|lm\rangle - \langle l'm'|\hat{V}_z\underbrace{\hat{L}_z|lm\rangle}_{=\hbar m|lm\rangle}$

so $\hbar(m'-m)\langle l'm'|\hat{V}_z|lm\rangle = 0$

then if $m'\neq m$, $\langle l'm'|\hat{V}_z|lm\rangle = 0$

Problem 4 (6.24)

1. $\hbar(m'-m)\langle l'm'|\hat{V}_z|lm\rangle = 0$ (from previous problem) (1)

2. $\langle l'm'|[\hat{L}_z,\hat{V}_\pm]|lm\rangle = \pm\langle l'm'|\hat{V}_\pm|lm\rangle$

$\hbar(m'-m)\langle l'm'|\hat{V}_\pm|lm\rangle = \pm\langle l'm'|\hat{V}_\pm|lm\rangle$

$\hbar(m'-m\mp 1)\langle l'm'|\hat{V}_\pm|lm\rangle = 0$ (2)

3. $\langle l'm'|[\hat{L}_+,\hat{V}_+]|lm\rangle = 0$

$B_{l'}^{m'}\langle l'm'-1|\hat{V}_+|lm\rangle - A_l^m\langle l'm'|\hat{V}_+|l\,m+1\rangle = 0$ (3)

$\langle l'm'|[\hat{L}_-,\hat{V}_-]|lm\rangle = 0$

$A_{l'}^{m'}\langle l'm'+1|\hat{V}_-|lm\rangle - B_l^m\langle l'm'|\hat{V}_-|l\,m-1\rangle = 0$ (4)

4. $\langle l'm'|[\hat{L}_+,\hat{V}_z]|lm\rangle = -\hbar\langle l'm'|\hat{V}_+|lm\rangle$

$B_{l'}^{m'}\langle l'm'-1|\hat{V}_z|lm\rangle - A_l^m\langle l'm'|\hat{V}_z|l\,m+1\rangle = -\hbar\langle l'm'|\hat{V}_+|lm\rangle$ (5)

$\langle l'm'|[\hat{L}_-,\hat{V}_z]|lm\rangle = \hbar\langle l'm'|\hat{V}_-|lm\rangle$

$A_{l'}^{m'}\langle l'm'+1|\hat{V}_z|lm\rangle - B_l^m\langle l'm'|\hat{V}_z|l\,m-1\rangle = \hbar\langle l'm'|\hat{V}_-|lm\rangle$ (6)

5. $\langle l'm'|[\hat{L}_+,\hat{V}_-]|lm\rangle = 2\hbar\langle l'm'|\hat{V}_z|lm\rangle$

$B_{l'}^{m'}\langle l'm'-1|\hat{V}_-|lm\rangle - A_l^m\langle l'm'|\hat{V}_-|l\,m+1\rangle = 2\hbar\langle l'm'|\hat{V}_z|lm\rangle$ (7)

$\langle l'm'|[\hat{L}_-,\hat{V}_+]|lm\rangle = -2\hbar\langle l'm'|\hat{V}_z|lm\rangle$

$A_{l'}^{m'}\langle l'm'+1|\hat{V}_+|lm\rangle - B_l^m\langle l'm'|\hat{V}_+|l\,m-1\rangle = -2\hbar\langle l'm'|\hat{V}_z|lm\rangle$ (8)

Problem 4 (continued)

From (3): $B_{\ell'}^{m'} \langle \ell' m'-1 | \hat{V}_+ | \ell m \rangle = A_\ell^m \langle \ell' m' | \hat{V}_+ | \ell\, m+1 \rangle$

" " assuming (6.59) is true

$$B_{\ell'}^{m'} \left(-\sqrt{2}\, \underset{\uparrow}{C_{m\,1\,m'-1}^{\ell 1 \ell'}} \alpha\right) = A_\ell^m \left(-\sqrt{2}\, C_{m+1\,1\,m'}^{\ell 1 \ell'} \alpha\right)$$

indep. of m, m'

we'll show that

$$B_{\ell'}^{m'} C_{m\,1\,m'-1}^{\ell 1 \ell'} = A_\ell^m C_{m+1\,1\,m'}^{\ell 1 \ell'}$$

(& hence (6.59)) using the recursion relation from (6.66)

from the lower recursion of (6.66)

$$B_{\ell'}^{m'} C_{m\,1\,m'-1}^{\ell 1 \ell'} = A_\ell^m C_{m+1\,1\,m'}^{\ell 1 \ell'} + \underbrace{A_1^1}_{=0} C_{m\,2\,m'}^{\ell 1 \ell'} = A_\ell^m C_{m+1\,1\,m'}^{\ell 1 \ell'} \checkmark$$

From (4): $A_{\ell'}^{m'} \langle \ell' m'+1 | \hat{V}_- | \ell m \rangle = B_\ell^m \langle \ell' m' | \hat{V}_- | \ell\, m-1 \rangle$

" " assuming (6.60) is true

$$A_{\ell'}^{m'} \left(\sqrt{2}\, C_{m-1\,m'+1}^{\ell 1 \ell'} \alpha\right) = B_\ell^m \left(\sqrt{2}\, C_{m-1\,-1\,m'}^{\ell 1 \ell'} \alpha\right)$$

so we need to show that

$$A_{\ell'}^{m'} C_{m\,-1\,m'+1}^{\ell 1 \ell'} = B_\ell^m C_{m-1\,-1\,m'}^{\ell 1 \ell'}$$

from the upper recursion in (6.66)

$$A_{\ell'}^{m'} C_{m\,-1\,m'+1}^{\ell 1 \ell'} = B_\ell^m C_{m-1\,-1\,m'}^{\ell 1 \ell'} + \underbrace{\cancel{B_1^{-1}}}_{0} C_{m\,-2\,m'}^{\ell 1 \ell'} \checkmark$$

From (5) $B_{\ell'}^{m'} \langle \ell' m'-1 | \hat{V}_z | \ell m \rangle - A_\ell^m \langle \ell' m' | \hat{V}_z | \ell\, m+1 \rangle = -\hbar \langle \ell' m' | \hat{V}_+ | \ell m \rangle$

" using (6.61) " " using (6.59)

$$B_{\ell'}^{m'} C_{m\,0\,m'-1}^{\ell 1 \ell'} \alpha - A_\ell^m C_{m+1\,0\,m'}^{\ell 1 \ell'} \alpha = -\hbar \cdot \alpha \left(-\sqrt{2}\, C_{m\,1\,m'}^{\ell 1 \ell'}\right)$$

we need to show that

$$B_{\ell'}^{m'} C_{m\,0\,m'-1}^{\ell 1 \ell'} = A_\ell^m C_{m+1\,0\,m'}^{\ell 1 \ell'} + \hbar\sqrt{2}\, C_{m\,1\,m'}^{\ell 1 \ell'}$$

from the lower recursion in (6.66):

$$B_{\ell'}^{m'} C_{m\,0\,m'-1}^{\ell 1 \ell'} = A_\ell^m C_{m+1\,0\,m'}^{\ell 1 \ell'} + \underbrace{A_1^0}_{= \hbar\sqrt{1 \cdot 2 - 0} = \hbar\sqrt{2}} C_{m\,1\,m'}^{\ell 1 \ell'} \checkmark$$

From (6) $A_{\ell'}^{m'}\langle \ell' m'+1|\hat{V}_z|\ell m\rangle - B_\ell^{m}\langle \ell' m'|\hat{V}_z|\ell m-1\rangle = \hbar\langle \ell' m'|\hat{V}_-|\ell m\rangle$

" using (6.61) " " using (6.60)

$$A_{\ell'}^{m'} C^{\ell 1 \ell'}_{m0m'+1}\,\alpha \quad - B_\ell^{m} C^{\ell 1 \ell'}_{m-10m'}\,\alpha = \hbar\left(\sqrt{2}\, C^{\ell 1 \ell'}_{m-1m'}\,\alpha\right)$$

we need to show that

$$A_{\ell'}^{m'} C^{\ell 1 \ell'}_{m0m'+1} = B_\ell^{m} C^{\ell 1 \ell'}_{m-10m'} + \hbar\sqrt{2}\, C^{\ell 1 \ell'}_{m-1m'}$$

the top relation in (6.66) gives:

$$A_{\ell'}^{m'} C^{\ell 1 \ell'}_{m0m'+1} = B_\ell^{m} C^{\ell 1 \ell'}_{m-10m'} + \underbrace{B_1^{0}}_{=\hbar\sqrt{1\cdot 2-0}\,=\,\hbar\sqrt{2}} C^{\ell 1 \ell'}_{m-1m'} \quad ✓$$

From (7) $B_{\ell'}^{m'}\langle \ell' m'-1|\hat{V}_-|\ell m\rangle - A_\ell^{m}\langle \ell' m'|\hat{V}_-|\ell m+1\rangle = 2\hbar\langle \ell' m'|\hat{V}_z|\ell m\rangle$

" using (6.60) " " using (6.61)

$$B_{\ell'}^{m'}\left(\sqrt{2}\, C^{\ell 1 \ell'}_{m-1m'-1}\,\alpha\right) - A_\ell^{m}\left(\sqrt{2}\, C^{\ell 1 \ell'}_{m+1-1m'}\,\alpha\right) = 2\hbar\left(\alpha\, C^{\ell 1 \ell'}_{m0m'}\right)$$

we need to show: $B_{\ell'}^{m'} C^{\ell 1 \ell'}_{m-1m'-1} = A_\ell^{m} C^{\ell 1 \ell'}_{m+1-1m'} + \hbar\sqrt{2}\, C^{\ell 1 \ell'}_{m0m'}$

the lower relation of (6.66) gives:

$$B_{\ell'}^{m'} C^{\ell 1 \ell'}_{m-1m'-1} = A_\ell^{m} C^{\ell 1 \ell'}_{m+1-1m'} + \underbrace{A_1^{-1}}_{=\hbar\sqrt{1\cdot 2-0}\,=\,\hbar\sqrt{2}} C^{\ell 1 \ell'}_{m0m'} \quad ✓$$

From (8) $A_{\ell'}^{m'}\langle \ell' m'+1|\hat{V}_+|\ell m\rangle - B_\ell^{m}\langle \ell' m'|\hat{V}_+|\ell m-1\rangle = -2\hbar\langle \ell' m'|\hat{V}_z|\ell m\rangle$

" using (6.59) " " using (6.61)

$$A_{\ell'}^{m'}\left(-\sqrt{2}\, C^{\ell 1 \ell'}_{m1m'+1}\,\alpha\right) - B_\ell^{m}\left(-\sqrt{2}\, C^{\ell 1 \ell'}_{m-11m'}\,\alpha\right) = -2\hbar\, C^{\ell 1 \ell'}_{m0m'}\,\alpha$$

we need to show: $A_{\ell'}^{m'} C^{\ell 1 \ell'}_{m1m'+1} = B_\ell^{m} C^{\ell 1 \ell'}_{m-11m'} + \hbar\sqrt{2}\, C^{\ell 1 \ell'}_{m0m'}$

the upper relation in (6.66) gives

$$A_{\ell'}^{m'} C^{\ell 1 \ell'}_{m1m'+1} = B_\ell^{m} C^{\ell 1 \ell'}_{m-11m'} + \underbrace{B_1^{1}}_{=\hbar\sqrt{1\cdot 2-0}\,=\,\hbar\sqrt{2}} C^{\ell 1 \ell'}_{m-10m'} \quad ✓$$

Problem 3 (6.21)

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|211\rangle + |21{-1}\rangle\right)$,

$$\langle\psi|r|\psi\rangle = \tfrac{1}{2}\Big(\underbrace{\langle 211|r|211\rangle}_{=\langle 21\|r\|21\rangle} + \underbrace{\langle 211|r|21{-1}\rangle}_{=0\ (1\neq -1)} + \underbrace{\langle 21{-1}|r|211\rangle}_{=0\ (1\neq -1)} + \underbrace{\langle 21{-1}|r|21{-1}\rangle}_{=\langle 21\|r\|21\rangle}\Big)$$

$$= \langle 21\|r\|21\rangle$$

The easiest state for computing $\langle 21\|r\|21\rangle$ is $|210\rangle$ w/

$$\psi_{210} = \frac{e^{-r/2a}}{\sqrt{24a^3}}\,\frac{r}{a}\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \ ,$$

$$\langle 210|r|210\rangle = \int_0^\infty\int_0^\pi\int_0^{2\pi} \frac{e^{-r/a}}{24a^3}\,\frac{r^2}{a^2}\,\frac{3}{4\pi}\cdot r\cdot\cos^2\theta\ r^2\sin\theta\,d\phi\,d\theta\,dr$$

$$= \frac{3}{48a^5}\int_0^\infty\int_0^\pi e^{-r/a}\,r^5\cos^2\theta\sin\theta\,d\theta\,dr \quad \theta\text{-int. gives } 2/3$$

$$= \frac{1}{24a^5}\underbrace{\int_0^\infty e^{-r/a}\,r^5\,dr}_{=120a^6} = 5a = \langle 21\|r\|21\rangle$$

so $\langle\psi|r|\psi\rangle = 5a$.

Problem 5 (6.25)

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|211\rangle + |200\rangle\right)$, +

$$\langle\psi|\vec{p}_e|\psi\rangle = \tfrac{1}{2}\Big(\underbrace{\langle 211|\vec{p}_e|211\rangle}_{=0,\ 1+1=2 \text{ Laporte}} + \langle 211|\vec{p}_e|200\rangle + \langle 200|\vec{p}_e|211\rangle + \underbrace{\langle 211|\vec{p}_e|211\rangle}_{=0\ (1+1=2) \text{ Laporte}}\Big) \quad \text{c.c.}$$

$$= \tfrac{1}{2}\left(2\,\mathrm{Re}(\langle 211|\vec{p}_e|200\rangle)\right) = \mathrm{Re}\left(\langle 211|\vec{p}_e|200\rangle\right) \quad (*)$$

we need to compute: $\langle 21\|p_e\|20\rangle$, & we'll use $\langle 210|qz|200\rangle$ (easiest choice)

$$\psi_{210} = \frac{e^{-r/2a}}{\sqrt{24a^3}}\,\frac{r}{a}\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \qquad \psi_{200} = \frac{e^{-r/2a}}{\sqrt{32\pi a^3}}(2-r/a) \qquad z = r\cos\theta$$

$$\langle 210|qz|200\rangle = q\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{e^{-r/a}}{\sqrt{24a^3}\sqrt{32\pi a^3}}\left(\frac{3}{4\pi}\right)^{1/2}\frac{r}{a}(2-r/a)\cos\theta\cdot r\cos\theta\cdot r^2\sin\theta\,d\phi\,d\theta\,dr$$

$$= -3qa = \underbrace{C^{011}_{000}}_{=1}\langle 21\|p_e\|20\rangle$$

Now: $\langle 211|p_z|200\rangle = 0$ (since $m'=1\neq m=0$), $\langle 211|\hat{p}_+|200\rangle = -\sqrt{2}\,\underbrace{C^{011}_{011}}_{=1}\langle 21\|p_e\|20\rangle$

$\langle 211|\hat{p}_+|200\rangle = -\sqrt{2}\,C^{011}_{011}\langle 21\|p_e\|20\rangle$, $\langle 211|\hat{p}_-|200\rangle = \sqrt{2}\,\underbrace{C^{011}_{0-10}}_{=0\ (0-1\neq 0)}\langle 21\|p_e\|20\rangle$

$\langle 211|\hat{p}_y|200\rangle$ is imaginary, so won't contribute to $(*)$ & $\langle 211|\hat{p}_x|200\rangle = -\tfrac{1}{2}\sqrt{2}\,\langle 21\|p_e\|20\rangle = \frac{3qa}{\sqrt{2}}$

$\langle\psi|\vec{p}_e|\psi\rangle = \frac{3qa}{\sqrt{2}}\,\hat{x}$, $q = -1.6\times10^{-19}$ C, $a = .5$ Å