Problem 1

Write: $f(x) = \underbrace{\tfrac{1}{2}(f(x)+f(-x))}_{=f_+(x)} + \underbrace{\tfrac{1}{2}(f(x)-f(-x))}_{=f_-(x)} = f_+(x) + f_-(x)$

we have: $\hat{\pi} f_+(x) = \tfrac{1}{2}(f(-x)+f(x)) = f_+(x)$ &

$\hat{\pi} f_-(x) = \tfrac{1}{2}(f(-x)-f(x)) = -f_-(x)$

so $f_+(x)$ & $f_-(x)$ are eigenfunctions of the parity operator.

Problem 2

Using $\hat{R}_z = e^{-i\varphi/\hbar \hat{L}_z} \approx (1 - i/\hbar\, \varphi \hat{L}_z)$, we have:

$$\hat{\vec{L}}' = (1 + i/\hbar\, \varphi \hat{L}_z)\hat{\vec{L}}(1 - i/\hbar\, \varphi \hat{L}_z)$$
$$\overset{!}{=} \hat{\vec{L}} + i/\hbar\, \varphi [\hat{L}_z, \hat{\vec{L}}] + O(\varphi^2)$$

We know that $[\hat{L}_z, \hat{L}_j] = i\hbar \epsilon_{zjk} \hat{L}_k$ so for $j=z$: $[\hat{L}_z, \hat{L}_y] = -i\hbar \hat{L}_x$ & $[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$, &

$$\hat{L}'_x = \hat{L}_x - \varphi \hat{L}_y \quad , \quad \hat{L}'_y = \hat{L}_y + \varphi \hat{L}_x \quad , \quad \hat{L}'_z = \hat{L}_z \quad \text{(inf. rot. about } \varphi)$$

& $\hat{L}$ transforms like a vector under rotations.

Problem 3 (6.13)

a. $\hat{\vec{p}}_e$ is a vector operator, & $\langle 100|\hat{\vec{p}}_e|100\rangle = 0$ since $l'=0$, $l=0$ & $l'+l$ is "even" (Laporte's rule).

b. we need to pick 2 states w/ $l'+l$ odd, try: $\langle 210|\hat{\vec{p}}_e|200\rangle$

working in position space:

$$\psi_{200} = \sqrt{\frac{1}{a^3 \cdot 8}}\, e^{-r/2a} L_1^1(r/a)\, Y_0^0(\theta,\phi) = \frac{e^{-r/2a}}{\sqrt{32\pi a^3}}(2 - r/a)$$

(4.89)

$$\psi_{210} = \sqrt{\frac{1}{a^3 \cdot 24}}\, e^{-r/2a} \frac{r}{a} \underbrace{L_0^3(r/a)}_{=1} Y_1^0(\theta,\phi) = \frac{e^{-r/2a}(r/a)}{\sqrt{24 a^3}}\left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

use $\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$ in

$\langle 210|q\vec{r}|200\rangle = \int_0^\infty \int_0^{2\pi} \int_0^\pi \psi_{210}^* q\vec{r}\, \psi_{200} \cdot r^2 \sin\theta\, d\theta\, d\phi\, dr$ – the $\hat{x}$ + $\hat{y}$ components go away via ϕ-int.

$$\overset{!}{=} \frac{\hat{z}}{16a^4}\int_0^\infty \int_0^\pi e^{-r/a} r^4 (2 - r/a)\cos^2\theta \sin\theta\, d\theta\, dr \quad \& \quad \int_0^\pi \cos^2\theta \sin\theta\, d\theta = 2/3$$

$$\overset{!}{=} \frac{\hat{z}}{24a^5}\underbrace{\int_0^\infty e^{-r/a}(2 - r/a) r^4 dr}_{=-72a^6} = -3a\hat{z}$$

Problem 4 (6.19)

For: $[\hat{L}_i, \hat{V}_j] = i\hbar\,\epsilon_{ijk}\hat{V}_k$, we have non-zero commutators:

$$[\hat{L}_x, \hat{V}_y] = i\hbar\hat{V}_z \qquad [\hat{L}_x, \hat{V}_z] = -i\hbar\hat{V}_y$$

$$[\hat{L}_y, \hat{V}_x] = -i\hbar\hat{V}_z \qquad [\hat{L}_y, \hat{V}_z] = i\hbar\hat{V}_x$$

$$[\hat{L}_z, \hat{V}_x] = i\hbar\hat{V}_y \qquad [\hat{L}_z, \hat{V}_y] = -i\hbar\hat{V}_x$$

a. $[\hat{L}_z, \hat{V}_\pm] = [\hat{L}_z, \hat{V}_x \pm i\hat{V}_y] = i\hbar\hat{V}_y \pm \hbar\hat{V}_x = \pm\hbar(\hat{V}_x \pm i\hat{V}_y) = \pm\hbar\hat{V}_\pm$

In order to get $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ commutators, we'll use $[A,B]=C \Rightarrow [A^2,B] = AC + CA$
so we need:

$$[\hat{L}_x, \hat{V}_\pm] = [\hat{L}_x, \hat{V}_x \pm i\hat{V}_y] = \mp\hbar\hat{V}_z$$

$$[\hat{L}_y, \hat{V}_\pm] = [\hat{L}_y, \hat{V}_x \pm i\hat{V}_y] = -i\hbar\hat{V}_z$$

and then:

$$[\hat{L}_x^2, \hat{V}_\pm] = \mp\hbar(\hat{L}_x\hat{V}_z + \hat{V}_z\hat{L}_x) = \mp\hbar(2\hat{V}_z\hat{L}_x - i\hbar\hat{V}_y) = \mp 2\hbar\hat{V}_z\hat{L}_x \pm i\hbar^2\hat{V}_y$$

$$[\hat{L}_y^2, \hat{V}_\pm] = -i\hbar(\hat{L}_y\hat{V}_z + \hat{V}_z\hat{L}_y) = -i\hbar(2\hat{V}_z\hat{L}_y + i\hbar\hat{V}_x) = -2i\hbar\hat{V}_z\hat{L}_y + \hbar^2\hat{V}_x$$

$$[\hat{L}_z^2, \hat{V}_\pm] = \pm\hbar(\hat{L}_z\hat{V}_\pm + \hat{V}_\pm\hat{L}_z) = \pm\hbar(2\hat{V}_\pm\hat{L}_z \pm \hbar\hat{V}_\pm) = \pm 2\hbar\hat{V}_\pm\hat{L}_z + \hbar^2\hat{V}_\pm$$

so: $[\hat{L}^2, \hat{V}_\pm] = [\hat{L}_x^2, \hat{V}_\pm] + [\hat{L}_y^2, \hat{V}_\pm] + [\hat{L}_z^2, \hat{V}_\pm]$ is

$$[\hat{L}^2, \hat{V}_\pm] = \mp 2\hbar\hat{V}_z\hat{L}_x \pm i\hbar^2\hat{V}_y - 2i\hbar\hat{V}_z\hat{L}_y + \hbar^2\hat{V}_x \pm 2\hbar\hat{V}_\pm\hat{L}_z + \hbar^2\hat{V}_\pm$$

$$= \mp 2\hbar\hat{V}_z\underbrace{(\hat{L}_x \pm i\hat{L}_y)}_{=\hat{L}_\pm} + \underbrace{\hbar^2(\hat{V}_x \pm i\hat{V}_y)}_{=\hbar^2\hat{V}_\pm} \pm 2\hbar\hat{V}_\pm\hat{L}_z + \hbar^2\hat{V}_\pm$$

$$= \mp 2\hbar\hat{V}_z\hat{L}_\pm + 2\hbar^2\hat{V}_\pm \pm 2\hbar\hat{V}_\pm\hat{L}_z$$

b. We are given: $\hat{L}^2|\psi\rangle = \hbar^2\ell(\ell+1)|\psi\rangle$ & $\hat{L}_z|\psi\rangle = \hbar\ell|\psi\rangle$ (top rung)
we want to evaluate the state: $\hat{V}_+|\psi\rangle$:

$$\hat{L}^2\hat{V}_+|\psi\rangle = (\hat{V}_+\hat{L}^2 - 2\hbar\hat{V}_z\hat{L}_+ + 2\hbar^2\hat{V}_+ + 2\hbar\hat{V}_+\hat{L}_z)|\psi\rangle$$ using the commutator from part a

$$= \hbar^2\ell(\ell+1)\hat{V}_+|\psi\rangle - 2\hbar\hat{V}_z\underbrace{\cancel{\hat{L}_+|\psi\rangle}}_{m=\ell} + 2\hbar^2\hat{V}_+|\psi\rangle + 2\hbar^2\ell\hat{V}_+|\psi\rangle$$

$$= \hbar^2(\ell(\ell+1) + 2(\ell+1))\hat{V}_+|\psi\rangle = \hbar^2(\ell+2)(\ell+1)\hat{V}_+|\psi\rangle$$

so $\hat{L}^2\hat{V}_+|\psi\rangle = \hbar^2(\ell+2)(\ell+1)\hat{V}_+|\psi\rangle$ & $\hat{V}_+|\psi\rangle$ is an e-state of $\hat{L}^2$ w/ e-val $\hbar^2(\ell+2)(\ell+1)$ (or $\hat{V}_+|\psi\rangle = 0$). - now checking $\hat{L}_z$:

$\hat{L}_z\hat{V}_+|\psi\rangle = (\hat{V}_+\hat{L}_z + \hbar\hat{V}_+)|\psi\rangle = \hbar(\ell+1)\hat{V}_+|\psi\rangle$ using the other commutator from part a.

& $\hat{V}_+|\psi\rangle$ is an eigenstate of $\hat{L}_z$ w/ e-val $\hbar(\ell+1)$ (or $\hat{V}_+|\psi\rangle = 0$)

Problem 5 (6.22c)

We had: $[\hat{L}_x, \hat{V}_y] = i\hbar \hat{V}_z$ $\quad [\hat{L}_x, \hat{V}_z] = -i\hbar \hat{V}_y$ $\quad$ from $[\hat{L}_i, \hat{V}_j] = i\hbar \epsilon_{ijk} \hat{L}_k$

$[\hat{L}_y, \hat{V}_x] = -i\hbar \hat{V}_z$ $\quad [\hat{L}_y, \hat{V}_z] = i\hbar \hat{V}_x$

$[\hat{L}_z, \hat{V}_x] = i\hbar \hat{V}_y$ $\quad [\hat{L}_z, \hat{V}_y] = -i\hbar \hat{V}_x$

(6.50) : $[\hat{L}_z, \hat{V}_z] = i\hbar \epsilon_{zzk} \hat{L}_k = 0$

(6.51) : done in previous problem.

(6.52): $[\hat{L}_\pm, \hat{V}_\pm] = [\hat{L}_x \pm i\hat{L}_y, \hat{V}_x \pm i\hat{V}_y]$

$= [\hat{L}_x, \pm i\hat{V}_y] \pm i[\hat{L}_y, \hat{V}_x] = \mp \hbar \hat{V}_z \pm \hbar \hat{V}_z = 0$

(6.53): $[\hat{L}_\pm, \hat{V}_z] = [\hat{L}_x \pm i\hat{L}_y, \hat{V}_z] = [\hat{L}_x, \hat{V}_z] \pm i[\hat{L}_y, \hat{V}_z] = -i\hbar \hat{V}_y \pm i(i\hbar \hat{V}_x)$

$= \mp \hbar(\hat{V}_x \pm i\hat{V}_y) = \mp \hbar \hat{V}_\pm$

(6.54): $[\hat{L}_\pm, \hat{V}_\pm] = [\hat{L}_x \pm i\hat{L}_y, \hat{V}_x \pm i\hat{V}_y] = \pm i[\hat{L}_x, \hat{V}_y] \pm i[\hat{L}_y, \hat{V}_x]$

$= \pm i(i\hbar \hat{V}_z - i\hbar \hat{V}_z) = 0$ ✓

Problem 6 (6.18)

a. Working w/ a test function: $[\hat{T}, \hat{\pi}] f(x) = \hat{T}\hat{\pi} f(x) - \hat{\pi}\hat{T} f(x)$

$= \hat{T} f(-x) - \hat{\pi} f(x-a)$

$= f(-x-a) - f(-x+a) \neq 0$

b. For $f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$, we have: $\hat{\pi} f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} = \frac{1}{\sqrt{2\pi\hbar}} e^{i(-p)x/\hbar} = f_{-p}(x)$

c. $\hat{T}(a)\left[\frac{1}{\sqrt{\pi\hbar}} \cos\left(\frac{px}{\hbar}\right)\right] = \frac{1}{\sqrt{\pi\hbar}} \cos\left(\frac{p(x-a)}{\hbar}\right) = \frac{1}{\sqrt{\pi\hbar}}\left(\cos\left(\frac{px}{\hbar}\right)\cos\left(\frac{pa}{\hbar}\right) + \sin\left(\frac{px}{\hbar}\right)\sin\left(\frac{pa}{\hbar}\right)\right)$

&

$\hat{T}(a)\left[\frac{1}{\sqrt{\pi\hbar}} \sin\left(\frac{px}{\hbar}\right)\right] = \frac{1}{\sqrt{\pi\hbar}} \sin\left(\frac{p(x-a)}{\hbar}\right) = \frac{1}{\sqrt{\pi\hbar}}\left(\sin(px/\hbar)\cos(pa/\hbar) - \cos(px/\hbar)\sin(pa/\hbar)\right)$

Problem 7

For $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + qV(t)\Psi$ w/ $\Psi(0,t) = 0 = \Psi(a,t)$

Moving the time-dependent potential over to the left, & taking $\Psi(x,t) = X(x)\,T(t)$:

$$i\hbar(\dot{T}X) - qV(t)TX = -\frac{\hbar^2}{2m}X''T \qquad \text{divide by } XT \text{ to get:}$$

$$i\hbar\,\dot{T}/T - qV(t) = -\frac{\hbar^2}{2m}X''/X$$

setting both sides equal to E, a constant, we get

$-\frac{\hbar^2}{2m}X''/X = E \Rightarrow X(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ w/ $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ as usual for the infinite sq. well.

For the t-eqn: $i\hbar\,\dot{T}/T - qV(t) = E \Rightarrow i\hbar\dot{T} = (E+qV(t))T$

or

$$\dot{T} = \frac{1}{i\hbar}(E+qV(t))T \qquad \text{let } w(t) \equiv \frac{1}{i\hbar}(E+qV(t))$$

$$\dot{T} = w(t)T \qquad (*)$$

Consider the function: $T(t) = \alpha e^{u(t)}$ for some $u(t)$ — what is the relation between $u(t)$ & $w(t)$? Putting into $(*)$

$$\dot{T} = \alpha\dot{u}e^{u} = wT = w\alpha e^{u} \Rightarrow \dot{u} = w \Rightarrow u(t) = \int_0^t w(\bar{t})\,d\bar{t}$$

So $T(t) = \alpha e^{\int_0^t w(\bar{t})d\bar{t}} = \alpha e^{\int_0^t \frac{1}{i\hbar}(E+qV(\bar{t}))d\bar{t}}$

$$= \alpha e^{-iEt/\hbar}\cdot e^{-i/\hbar\int_0^t qV(\bar{t})d\bar{t}}$$

The solution for $\Psi(x,t)$ is: (adding up all the E_n contributions)

$$\Psi(x,t) = \sum_{j=1}^{\infty} c_j e^{-iE_jt/\hbar}\cdot e^{-i/\hbar\int_0^t qV(\bar{t})d\bar{t}}\cdot\sqrt{\frac{2}{a}}\sin\left(\frac{j\pi x}{a}\right)$$

$$= e^{-i/\hbar\int_0^t qV(\bar{t})d\bar{t}}\underbrace{\sum_{j=1}^{\infty} c_j e^{-iE_jt/\hbar}\sqrt{\frac{2}{a}}\sin\left(\frac{j\pi x}{a}\right)}_{\equiv\bar{\Psi}(x,t)}$$

& $\Psi(x,t) = \underbrace{e^{-i/\hbar\int_0^t qV(\bar{t})d\bar{t}}}_{\text{undetectable, time-varying pure phase}}\bar{\Psi}(x,t)$