Problem 1 | | Problem Set 8

For $\mathbb{A}\vec{v}_i = \lambda_i \vec{v}_i$, mult. by $\mathbb{B}$: $\mathbb{B}\mathbb{A}\vec{v}_i = \lambda_i \mathbb{B}\vec{v}_i$ & use $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$

$$\mathbb{A}(\mathbb{B}\vec{v}_i) = \lambda_i(\mathbb{B}\vec{v}_i) \Rightarrow \mathbb{B}\vec{v}_i \parallel \vec{v}_i$$

so $\mathbb{B}\vec{v}_i = \sigma_i \vec{v}_i$ & $\vec{v}_i$ is also an eigenvector of $\mathbb{B}$.

Problem 2

For $L_i = \sum_{\ell,k=1}^{3} \epsilon_{i\ell k} x_\ell p_k$, we have:

$$\{L_i, p_j\}_{PB} = \sum_n \left( \frac{\partial L_i}{\partial x_n} \underbrace{\frac{\partial p_j}{\partial p_n}}_{=\delta_{jn}} - \frac{\partial L_i}{\partial p_n} \cancel{\frac{\partial p_j}{\partial x_n}} \right)$$

$$= \frac{\partial L_i}{\partial x_j} = \sum_{\ell,k=1}^{3} \epsilon_{i\ell k} \underbrace{\frac{\partial x_\ell}{\partial x_j}}_{=\delta_{j\ell}} p_k = \epsilon_{ijk} p_k$$

then the commutator, obtained via $\{,\}_{PB}\, i\hbar \longrightarrow [\,,\,]$ is

$$[\hat{L}_i, \hat{p}_j] = i\hbar\, \epsilon_{ijk} \hat{p}_k$$

Problem 3

Take $\mathbb{A} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\mathbb{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, w/ $[\mathbb{A},\mathbb{B}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

see attached for the calculation of $e^{\mathbb{A}}$, $e^{\mathbb{B}}$ & $e^{\mathbb{A}+\mathbb{B}}$ to establish that:

$$e^{\mathbb{A}} e^{\mathbb{B}} \neq e^{\mathbb{A}+\mathbb{B}}$$

# Problem 3

Decompose A, B, and A+ B into eigenvalue and eigenvector matrices for exponentiation:

In[1]:=
```
A = {{1, 0}, {0, 2}};
B = {{0, 1}, {1, 0}};
W = A + B;
```

In[4]:=
```
DA = DiagonalMatrix[{2, 1}];
VA = {{0, 1}, {1, 0}};

DB = DiagonalMatrix[{-1, 1}];
VB = {{-1, 1}, {1, 1}};
```

$$DW = \text{DiagonalMatrix}\left[\left\{\frac{1}{2}\left(3+\sqrt{5}\right), \frac{1}{2}\left(3-\sqrt{5}\right)\right\}\right];$$

$$VW = \left\{\left\{\frac{1}{2}\left(-1+\sqrt{5}\right), \frac{1}{2}\left(-1-\sqrt{5}\right)\right\}, \{1, 1\}\right\};$$

Check them

In[10]:=
```
A - VA.DA.Inverse[VA]
```

Out[10]=
```
{{0, 0}, {0, 0}}
```

In[11]:=
```
B - VB.DB.Inverse[VB]
```

Out[11]=
```
{{0, 0}, {0, 0}}
```

In[12]:=
```
Simplify[W - VW.DW.Inverse[VW]]
```

Out[12]=
```
{{0, 0}, {0, 0}}
```

In[13]:=
```
ExpA = VA.{{Exp[DA[[1, 1]]], 0}, {0, Exp[DA[[2, 2]]]}}.Inverse[VA]
```

Out[13]=

$$\{\{e, 0\}, \{0, e^2\}\}$$

In[14]:=
```
ExpB = VB.{{Exp[DB[[1, 1]]], 0}, {0, Exp[DB[[2, 2]]]}}.Inverse[VB]
```

Out[14]=

$$\left\{\left\{\frac{1}{2e}+\frac{e}{2}, -\frac{1}{2e}+\frac{e}{2}\right\}, \left\{-\frac{1}{2e}+\frac{e}{2}, \frac{1}{2e}+\frac{e}{2}\right\}\right\}$$

In[15]:= 

```
ExpW = VW.{{Exp[DW[[1, 1]]], 0}, {0, Exp[DW[[2, 2]]]}}.Inverse[VW]
```

Out[15]=

$$\left\{\left\{-\frac{\left(-1-\sqrt{5}\right) e^{\frac{1}{2}\left(3-\sqrt{5}\right)}}{2\sqrt{5}}+\frac{\left(-1+\sqrt{5}\right) e^{\frac{1}{2}\left(3+\sqrt{5}\right)}}{2\sqrt{5}},\ \frac{\left(-1-\sqrt{5}\right)\left(-1+\sqrt{5}\right) e^{\frac{1}{2}\left(3-\sqrt{5}\right)}}{4\sqrt{5}}+\frac{\left(-1+\sqrt{5}\right)\left(1+\sqrt{5}\right) e^{\frac{1}{2}\left(3+\sqrt{5}\right)}}{4\sqrt{5}}\right\},\ \left\{-\frac{e^{\frac{1}{2}\left(3-\sqrt{5}\right)}}{\sqrt{5}}+\frac{e^{\frac{1}{2}\left(3+\sqrt{5}\right)}}{\sqrt{5}},\ \frac{\left(-1+\sqrt{5}\right) e^{\frac{1}{2}\left(3-\sqrt{5}\right)}}{2\sqrt{5}}+\frac{\left(1+\sqrt{5}\right) e^{\frac{1}{2}\left(3+\sqrt{5}\right)}}{2\sqrt{5}}\right\}\right\}$$

In[16]:= 

```
ExpA.ExpB
```

Out[16]=

$$\left\{\left\{\left(\frac{1}{2e}+\frac{e}{2}\right)e,\ \left(-\frac{1}{2e}+\frac{e}{2}\right)e\right\},\ \left\{\left(-\frac{1}{2e}+\frac{e}{2}\right)e^2,\ \left(\frac{1}{2e}+\frac{e}{2}\right)e^2\right\}\right\}$$

This doesn’t look like the exponential of W = A + B:

In[17]:= 

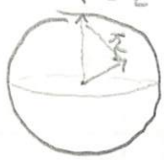
```
Simplify[ExpW - ExpA.ExpB]
```

Out[17]=

$$\left\{\left\{\frac{1}{10}\left(-5-5e^2+\left(5+\sqrt{5}\right)e^{\frac{3}{2}-\frac{\sqrt{5}}{2}}-\left(-5+\sqrt{5}\right)e^{\frac{1}{2}\left(3+\sqrt{5}\right)}\right),\ \frac{1}{2}-\frac{e^2}{2}-\frac{e^{\frac{3}{2}-\frac{\sqrt{5}}{2}}}{\sqrt{5}}+\frac{e^{\frac{1}{2}\left(3+\sqrt{5}\right)}}{\sqrt{5}}\right\},\ \left\{\frac{e}{2}-\frac{e^3}{2}-\frac{e^{\frac{3}{2}-\frac{\sqrt{5}}{2}}}{\sqrt{5}}+\frac{e^{\frac{1}{2}\left(3+\sqrt{5}\right)}}{\sqrt{5}},\ \frac{1}{10}\left(-5e-5e^3-\left(-5+\sqrt{5}\right)e^{\frac{3}{2}-\frac{\sqrt{5}}{2}}+\left(5+\sqrt{5}\right)e^{\frac{1}{2}\left(3+\sqrt{5}\right)}\right)\right\}\right\}$$

They are not the same.

# Problem 4




$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ then $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = x\hat{x} + y\hat{y} + (z-1)\hat{z}$

The vector $\vec{w}(t) = \vec{r}\,' + \vec{\mathscr{r}}\,t$ points from $\vec{r}\,'$ to $\vec{r}$ w/ parameter $t$. - at $t=0$, $\vec{w}(0) = \vec{r}\,'$, & at $t=1$, $\vec{w}(1) = \vec{r}$

We want $t^* \ni \vec{w}(t^*)\cdot\hat{z} = 0 \Rightarrow (1 + (z-1)t^*) = 0 \Rightarrow t^* = \dfrac{1}{1-z}$

& $\vec{w}(t^*) = \hat{z} + (x\hat{x} + y\hat{y} + (z-1)\hat{z})\cdot\dfrac{1}{1-z}$

$$= \frac{x}{1-z}\hat{x} + \frac{y}{1-z}\hat{y}$$

& if we associate $\hat{x}$ w/ the Real axis, $\hat{y}$ w/ the imaginary axis, then the stereo point is at

$$u = \frac{x}{1-z} + \frac{iy}{1-z}$$

to recover $x, y$ & $z$, note that: $u^*u = \dfrac{x^2+y^2}{(1-z)^2}$ & $x^2+y^2+z^2=1 \Rightarrow x^2+y^2 = (1-z^2)$

$$= \frac{(1-z)(1+z)}{(1-z)(1-z)} = \frac{1+z}{1-z}$$

so $(1-z)|u|^2 = 1+z \Rightarrow z(1+|u|^2) = |u|^2 - 1 \Rightarrow z = \dfrac{|u|^2-1}{1+|u|^2}$.

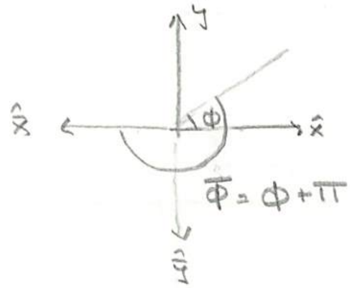
w/ $u + u^* = \dfrac{2x}{1-z}$ & $1 - z = \dfrac{1+|u|^2-|u|^2+1}{1+|u|^2} = \dfrac{2}{1+|u|^2}$ so $x = \dfrac{u+u^*}{1+uu^*}$

$u - u^* = \dfrac{2iy}{1-z} \Rightarrow y = \dfrac{u-u^*}{i(1+uu^*)}$ & $z = \dfrac{uu^*-1}{uu^*+1}$

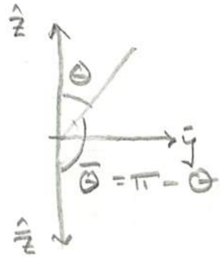
## Problem 5

Under axes-inversion, $r$, the distance to the origin, is unchanged : $\bar{r} = r$

For $\phi$, the angle measured w.r.t the $\hat{x}$-axis:




For the polar angle $\theta$




## Problem 6 (6.8)

a. $\langle f | \hat{\pi} | g \rangle = \int_{-\infty}^{+\infty} f(x)^* \hat{\pi}\, g(x)\, dx = \int_{-\infty}^{+\infty} f(x)^* g(-x)\, dx = -\int_{\infty}^{-\infty} f^*(-y)\, g(y)\, dy = \int_{-\infty}^{+\infty} f^*(-y)\, g(y)\, dy$ ($y = -x$)

$= \int_{-\infty}^{+\infty} (\hat{\pi} f(y))^* g(y)\, dy = \langle \hat{\pi} f | g \rangle$ so $\langle f | \hat{\pi}^\dagger | g \rangle = \langle f | \hat{\pi} | g \rangle$

$\Rightarrow \hat{\pi}^\dagger = \hat{\pi}$

b. for a function $f(x)$ w/ $\hat{\pi} f(x) = \lambda f(x)$, we have, acting w/ $\hat{\pi}$ again:

$$\hat{\pi}(\hat{\pi} f(x)) = \lambda \hat{\pi}(x) f(x)$$

$$f(x) = \lambda^2 f(x) \to \lambda = \pm 1$$

## Problem 7 (6.9)

a. $\hat{\pi}^\dagger f \hat{\pi} = f \Rightarrow f\hat{\pi} - \hat{\pi} f = 0 \Rightarrow [f, \hat{\pi}] = 0$ true scalar

($\hat{\pi}^{-1} = \hat{\pi}^\dagger = \hat{\pi}$)

$\hat{\pi}^\dagger f \hat{\pi} = -f \Rightarrow f\hat{\pi} + \hat{\pi} f = 0 \Rightarrow \{f, \hat{\pi}\} = 0$ pseudo scalar

b. $\hat{\pi}^\dagger \hat{\vec{V}} \hat{\pi} = -\hat{\vec{V}} \Rightarrow \hat{\vec{V}}\hat{\pi} + \hat{\pi}\hat{\vec{V}} = 0 \Rightarrow \{\hat{\vec{V}}, \hat{\pi}\} = 0$ true vector

$\hat{\pi}^\dagger \hat{\vec{V}} \hat{\pi} = +\hat{\vec{V}} \Rightarrow \hat{\vec{V}}\hat{\pi} - \hat{\pi}\hat{\vec{V}} = 0 \Rightarrow [\hat{\vec{V}}, \hat{\pi}] = 0$ pseudovectors