## Problem 1

## Problem Set 2

$$e^{i\mathbb{A}} = \sum_{j=0}^{\infty} \frac{1}{j!}(i\mathbb{A})^j$$

& for $\mathbb{A} = \mathbb{V}\mathbb{D}\mathbb{V}^{+}$, $\mathbb{A}^2 = \mathbb{V}\mathbb{D}\underbrace{\mathbb{V}^{+}\mathbb{V}}_{=\mathbb{I}}\mathbb{D}\mathbb{V}^{+} = \mathbb{V}\mathbb{D}^2\mathbb{V}^{+}$

& then $\mathbb{A}^j = \mathbb{V}\mathbb{D}^j\mathbb{V}^{+}$, so

$$= \mathbb{V}\left[\sum_{j=0}^{\infty} \frac{1}{j!}(i\mathbb{D})^j\right]\mathbb{V}^{+} = \mathbb{V}e^{i\mathbb{D}}\mathbb{V}^{+}$$

$$\begin{pmatrix} e^{i\lambda_1} & & \\ & e^{i\lambda_2} & \\ & & \ddots \, e^{i\lambda_n} \end{pmatrix}$$

Then for $\mathbb{U} = e^{i\mathbb{A}} = \mathbb{V}e^{i\mathbb{D}}\mathbb{V}^{+}$, we have

$$\mathbb{U}^{+} = (\mathbb{V}e^{i\mathbb{D}}\mathbb{V}^{+})^{+} = \mathbb{V}e^{-i\mathbb{D}}\mathbb{V}^{+} = \mathbb{V}\left[\sum_{j=0}^{\infty}\frac{1}{j!}(-i\mathbb{D})^j\right]\mathbb{V}^{+} = e^{-i\mathbb{A}}$$

$$= \begin{pmatrix} e^{-i\lambda_1} & & \\ & e^{-i\lambda_2} & \\ & & \ddots \, e^{-i\lambda_n} \end{pmatrix}$$

and $\mathbb{U}^{+}\mathbb{U} = \mathbb{V}e^{-i\mathbb{D}}\mathbb{V}^{+}\mathbb{V}e^{i\mathbb{D}}\mathbb{V}^{+} = \mathbb{V}\underbrace{e^{-i\mathbb{D}}e^{i\mathbb{D}}}_{=\mathbb{I}}\mathbb{V}^{+} = \mathbb{V}\mathbb{V}^{+} = \mathbb{I}$

## Problem 2

The eigenvalue problem for $\mathbb{L}_z$: $\mathbb{L}_z\mathbb{V} = \mathbb{V}\mathbb{D}$ has: $\mathbb{D} \doteq \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$

& $\mathbb{V} = \begin{pmatrix} i & -i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then $\mathbb{L}_z = \mathbb{V}\mathbb{D}\mathbb{V}^{-1}$ w/ $\mathbb{V}^{-1} \doteq \begin{pmatrix} -i/2 & 1/2 & 0 \\ i/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

As in the first problem, we have: $\mathbb{L}_z^j = \mathbb{V}\mathbb{D}^j\mathbb{V}^{-1}$, so that

$$e^{\Theta\mathbb{L}_z} = \sum_{j=0}^{\infty}\frac{(\Theta\mathbb{L}_z)^j}{j!} = \mathbb{V}\left[\sum_{j=0}^{\infty}\frac{1}{j!}(\Theta\mathbb{D})^j\right]\mathbb{V}^{-1} = \mathbb{V}e^{\Theta\mathbb{D}}\mathbb{V}^{-1}$$

w/ $e^{\Theta\mathbb{D}} \doteq \begin{pmatrix} e^{i\Theta} & 0 & 0 \\ 0 & e^{-i\Theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so $e^{\Theta\mathbb{L}_z} = \mathbb{V}e^{\Theta\mathbb{D}}\mathbb{V}^{-1} = \begin{pmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Problem 3

a. $-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = i\hbar\frac{\partial\psi}{\partial t} \Rightarrow \frac{\partial\psi}{\partial t} = \frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi\right]$ +

$$\frac{\partial\psi^*}{\partial t} = -\frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi^* + U\psi^*\right]$$

Now, $\frac{d}{dt}\int_\Omega \psi^*\psi\, d\tau = \int_\Omega\left[\frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}\right]d\tau$

$$= \int_\Omega\left\{-\frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi^*\psi + U\psi^*\psi\right] + \frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi\psi^* + U\psi^*\psi\right]\right\}d\tau$$

$$= \frac{1}{i\hbar}\int_\Omega\left[\frac{\hbar^2}{2m}\nabla^2\psi^*\psi - \frac{\hbar^2}{2m}\nabla^2\psi\psi^*\right]d\tau \qquad (*)$$

Note that. $\int_\Omega \nabla\cdot(\nabla\psi^*\psi)\,d\tau = \oint_{\partial\Omega}(\nabla\psi^*\cdot\psi)\cdot d\vec{a}$ by the div. thm.

|| (product rule)

$$\int_\Omega\left[\nabla^2\psi^*\psi + \nabla\psi^*\cdot\nabla\psi\right]d\tau = \int_{\partial\Omega}\left[\nabla\psi^*\psi\right]\cdot d\vec{a}$$

then $\int_\Omega \nabla^2\psi^*\psi\,d\tau = -\int_\Omega\nabla\psi^*\cdot\nabla\psi\,d\tau + \int_{\partial\Omega}\left[\nabla\psi^*\psi\right]\cdot d\vec{a}$

& sim. $\int_\Omega \nabla^2\psi\psi^*\,d\tau = -\int_\Omega\nabla\psi\cdot\nabla\psi^*\,d\tau + \int_{\partial\Omega}\left[\nabla\psi\psi^*\right]\cdot d\vec{a}$

Using these back in $(*)$,

$$\frac{d}{dt}\int_\Omega\psi^*\psi\,d\tau = \frac{\hbar}{i2m}\int_{\partial\Omega}\left[\psi\nabla\psi^* - \psi^*\nabla\psi\right]\cdot d\vec{a}$$

& $\vec{J} = -\frac{\hbar}{2mi}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

b. If $\Omega$ is all space, $-\int_{\partial\Omega}\vec{J}\cdot d\vec{a} = 0$ since $\psi \xrightarrow[r\to\infty]{} 0$, where the integral is evaluated.

Problem 4

For $\psi = Re^{iS}$, $\frac{\partial\psi}{\partial t} = \dot{R}e^{iS} + i\dot{S}Re^{iS} = \left(\frac{\dot{R}}{R} + i\dot{S}\right)\psi$

& $\nabla\psi = \nabla R e^{iS} + Re^{iS} i\nabla S \qquad \nabla\psi^* = \nabla R e^{-iS} - Re^{-iS} i \nabla S$

w/ $\nabla^2\psi = \nabla^2 R e^{iS} + i\nabla R\cdot\nabla S e^{iS} + i\nabla R\cdot\nabla S e^{iS} + i\nabla^2 S Re^{iS} - R\nabla S\cdot\nabla S e^{iS}$

$= \left[\frac{\nabla^2 R}{R} + 2\frac{i\nabla R\cdot\nabla S}{R} + i\nabla^2 S - \nabla S\cdot\nabla S\right]\psi$

then $-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = i\hbar\frac{\partial\psi}{\partial t}$ becomes

$$-\frac{\hbar^2}{2m}\left[\frac{\nabla^2 R}{R} + i\left(2\frac{\nabla R\cdot\nabla S}{R} + \nabla^2 S\right) - \nabla S\cdot\nabla S\right]\psi + U\psi = \left(\frac{i\hbar\dot{R}}{R} - \hbar\dot{S}\right)\psi$$

the real part of this eqn. is:

$$-\frac{\hbar^2}{2m}\left[\frac{\nabla^2 R}{R} - \nabla S\cdot\nabla S\right] + U = -\hbar\dot{S}$$

w/ imaginary part: $-\frac{\hbar^2}{2m}\left[\frac{2\nabla R\cdot\nabla S}{R} + \nabla^2 S\right] = \hbar\frac{\dot{R}}{R}$

The probability density is: $\rho = \psi^*\psi = R^2$

$\vec{J} = K[\psi\nabla\psi^* - \psi^*\nabla\psi]$

$= K\left[Re^{iS}(\nabla R e^{-iS} - Re^{-iS} i\nabla S) - Re^{-iS}(\nabla R e^{iS} + Re^{iS} i\nabla S)\right]$

$= K[R\nabla R - iR^2\nabla S - R\nabla R - iR^2\nabla S]$

$= -2iKR^2\nabla S \quad$ w/ $K = \frac{i\hbar}{2m}$,

$\vec{J} = \hbar/m \, R^2\nabla S$

## Problem 5

We have $\frac{dJ}{dt} = \{J, H\}$ for functions $J$ & $H$. The rule to get QM ops is $\{,\} \to \frac{1}{i\hbar}[,]$, so $i\hbar \frac{d\hat{J}}{dt} = [\hat{J}, \hat{H}]$

& taking the expectation value of both sides,

$$\frac{d\langle \hat{J} \rangle}{dt} = \frac{1}{i\hbar}\langle [\hat{J}, \hat{H}] \rangle.$$

## Problem 6

Taking $\hat{J} = \vec{r}$, we need to compute: $[\vec{r}, \hat{H}]$ w/:

$$\hat{H} = (p^2 - q\vec{p}\cdot\vec{A} - q\vec{A}\cdot\vec{p} + q^2A^2)\cdot\frac{1}{2m}$$

the first commutator is (take the x-component to get the pattern)

$$[x, p_x^2 + p_y^2 + p_z^2] = [x, p_x^2] = xp_xp_x - p_xp_xx = (i\hbar + p_xx)p_x - p_x(xp_x - i\hbar) = 2i\hbar p_x$$

& sim. for $[y, p^2] = 2i\hbar p_y$, $[z, p^2] = 2i\hbar p_z$

$$[x, p_xA_x + p_yA_y + p_zA_z] = [x, p_xA_x] = xp_xA_x - p_xA_xx = (p_xx + i\hbar)A_x - p_xxA_x = i\hbar A_x$$

w/ $[y, \vec{p}\cdot\vec{A}] = i\hbar A_y$ & $[z, \vec{p}\cdot\vec{A}] = i\hbar A_z$

Then we have:

$$[x, A_xp_x + A_yp_y + A_zp_z] = [x, A_xp_x] = xA_xp_x - A_xp_xx = A_xxp_x - A_x(xp_x - i\hbar) = i\hbar A_x$$

w/ $[y, \vec{A}\cdot\vec{p}] = i\hbar A_y$, $[z, \vec{A}\cdot\vec{p}] = i\hbar A_z$.

Then: $[x, \hat{H}] = \frac{1}{2m}\left([x, p^2] - q[x, \vec{p}\cdot\vec{A}] - q[x, \vec{A}\cdot\vec{p}] + q^2[x, A^2]\right)$

$$= \frac{1}{2m}(2i\hbar p_x - 2i\hbar qA_x) = i\hbar\frac{1}{m}(p_x - qA_x)$$

$\to \quad [\vec{r}, \hat{H}] = i\hbar\frac{1}{m}(\vec{p} - q\vec{A})$

Ehrenfest's theorem says: $\frac{d\langle \vec{r} \rangle}{dt} = \frac{1}{i\hbar}\langle [\vec{r}, \hat{H}] \rangle$

$$\frac{d\langle \vec{r} \rangle}{dt} = \frac{1}{m}\langle \vec{p} - q\vec{A} \rangle \checkmark$$