The Hamiltonian is: $H = \frac{1}{2m}(p_x^2 + p_z^2) + U_0\Theta(z)$ has $\dot{p}_x = -\frac{\partial H}{\partial x} = 0$

so that $p_x = mV_x$ is a constant of the motion, $V_x = V\sin\theta_i$.

For $z<0$ $E = \frac{1}{2}mV^2$, also a constant of the motion — then for $z>0$,

$$E = \tfrac{1}{2}mV_x^2 + \tfrac{1}{2}mV_z^2 + U_0 \Rightarrow \tfrac{1}{2}mV^2 = \tfrac{1}{2}mV^2\sin^2\theta_i + \tfrac{1}{2}mV_z^2 + U_0$$

$$\rightarrow \quad V_z = \left[V^2\cos^2\theta_i - \tfrac{2}{m}U_0\right]^{1/2}$$

The final velocity vector is: $\vec{V}_f = V_f\sin\theta_f\,\hat{x} + V_f\cos\theta_f\,\hat{z}$

$$= V\sin\theta_i\,\hat{x} + \left[V^2\cos^2\theta_i - \tfrac{2}{m}U_0\right]^{1/2}\hat{z}$$

giving
$$\tan\theta_f = \frac{\sin\theta_i}{\cos\theta_i\left[1 - \frac{2U_0}{mV^2\cos^2\theta_i}\right]^{1/2}} = \frac{\tan\theta_i}{\left[1 - \frac{2U_0}{mV^2\cos^2\theta_i}\right]^{1/2}} \Rightarrow \frac{\sin\theta_f}{\sin\theta_i} = \frac{1}{\sqrt{1 - \frac{U_0}{\frac{1}{2}mV^2}}}$$

Problem 2

$$L = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}mr^2\sin^2\theta\,\dot{\phi}^2 - U$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = p_r/m, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi} \Rightarrow \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$$

$$H = \left(\dot{r}p_r + \dot{\theta}p_\theta + \dot{\phi}p_\phi - L\right) = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{mr^2\sin^2\theta} - \left[\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} - U\right]$$

$$= \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right] + U(r,\theta,\phi)$$

If U is ϕ-indep. then $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \rightarrow p_\phi = mr^2\sin^2\theta\,\dot{\phi}$ is a constant of the motion.

This expression has units of angular momentum

Problem 3

$$\{x,p\} = \frac{\partial x}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial x}{\partial p}\frac{\partial p}{\partial x} = 1$$

$$\rightarrow [x,p] = i\hbar, \text{ so } [x,p] = i\hbar\{x,p\}$$

Problem 4

$$\frac{\partial J}{\partial t} = \frac{\partial J}{\partial x}\dot{x} + \frac{\partial J}{\partial p}\dot{p} + \frac{\partial J}{\partial t} \quad \& \quad \dot{x} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial x}$$

$$= \frac{\partial J}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial J}{\partial p}\frac{\partial H}{\partial x} + \frac{\partial J}{\partial t} = \{J, H\} + \frac{\partial J}{\partial t}.$$

Problem 5

$$H = \frac{1}{2m}(\vec{p}-q\vec{A})\cdot(\vec{p}-q\vec{A}) = \frac{1}{2m}\left[p_x^2+p_y^2+p_z^2-2q(p_xA_x+p_yA_y+p_zA_z)+q^2(A_x^2+A_y^2+A_z^2)\right]$$

I'll work out the x eqns carefully, y & z are similar:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}\left[p_x - qA_x\right] \Rightarrow p_x = m\dot{x} + qA_x \quad \& \text{ sim. } p_y = m\dot{y}+qA_y,\ p_z = m\dot{z}+qA_z.$$

we have $\vec{p} = m\vec{v} + q\vec{A}$.

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\left[-\frac{q}{m}\vec{p}\cdot\frac{\partial\vec{A}}{\partial x} + \frac{q^2}{m}\vec{A}\cdot\frac{\partial \vec{A}}{\partial x}\right] \qquad (*)$$

& $\dot{\vec{p}} = m\dot{\vec{v}} + q\dot{\vec{A}}$

For $\vec{A}$ a function of position, $\dot{\vec{A}} = \frac{\partial\vec{A}}{\partial x}\dot{x} + \frac{\partial\vec{A}}{\partial y}\dot{y} + \frac{\partial\vec{A}}{\partial z}\dot{z}$,

The LHS of $(*)$ is:

$\dot{p}_x = m\ddot{x} + q\vec{v}\cdot(\nabla A_x)$, & the RHS is:

$$\frac{q}{m}\vec{p}\cdot\frac{\partial\vec{A}}{\partial x} - \frac{q^2}{m}\vec{A}\cdot\frac{\partial\vec{A}}{\partial x} = \frac{q}{m}\left[\vec{p}-q\vec{A}\right]\cdot\frac{\partial\vec{A}}{\partial x} = q\vec{v}\cdot\frac{\partial\vec{A}}{\partial x}$$

so we have:

$$m\ddot{x} + q\vec{v}\cdot(\nabla A_x) = q\vec{v}\cdot\frac{\partial\vec{A}}{\partial x} \Rightarrow m\ddot{x} = q\vec{v}\cdot\left[\frac{\partial\vec{A}}{\partial x} - \nabla A_x\right]$$

w/ $m\ddot{y} = q\vec{v}\cdot\left[\frac{\partial\vec{A}}{\partial y} - \nabla A_y\right]$ & $m\ddot{z} = q\vec{v}\cdot\left[\frac{\partial\vec{A}}{\partial z} - \nabla A_z\right]$

For the terms in brackets: $\frac{\partial\vec{A}}{\partial x} - \nabla A_x = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{y} + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{z}$

$$= (\nabla\times\vec{A})_z\,\hat{y} - (\nabla\times\vec{A})_y\,\hat{z}$$

$$\& \ \vec{v}\cdot\left[\frac{\partial\vec{A}}{\partial x} - \nabla A_x\right] = v_y\underbrace{(\nabla\times\vec{A})_z}_{=B_z} - v_z\underbrace{(\nabla\times\vec{A})_y}_{=B_y} = v_yB_z - v_zB_y = (\vec{v}\times\vec{B})_x$$

and $m\ddot{x} = q(\vec{v}\times\vec{B})_x$ & sim. for y & z, then

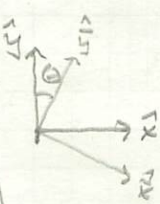
$$m\ddot{\vec{r}} = q\,\dot{\vec{r}}\times\vec{B} \quad \text{w/ } \vec{B} = \nabla\times\vec{A}.$$

Problem 6

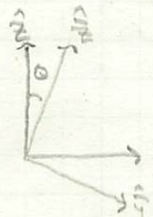
$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{w/ } \cos\theta \approx 1,\ \sin\theta \approx \theta$$

$$\approx \Bigg[\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{= \mathbb{I}} + \underbrace{\begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{= \theta \mathbb{L}_z} \Bigg] \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{so } \mathbb{L}_z \doteq \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

looking down the z-axis, we are rotating the axes clockwise by θ:



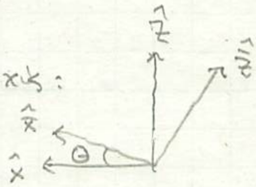
looking down the $\hat{x}$ axis:



$$\text{so } \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{w/ } \mathbb{L}_x \doteq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

looking down the $\hat{y}$ axis:



$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{+ } \mathbb{L}_y \doteq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

The commutators (see attached) are $[\mathbb{L}_x, \mathbb{L}_y] = \mathbb{L}_z$

$$[\mathbb{L}_x, \mathbb{L}_z] = -\mathbb{L}_y$$

$$[\mathbb{L}_y, \mathbb{L}_z] = \mathbb{L}_x$$

or $[\mathbb{L}_i, \mathbb{L}_j] = \epsilon_{ijk} \mathbb{L}_k$ "just like" angular momentum operators (except for the factor of $i\hbar$...)

In[1]:=
```
Lx = {{0, 0, 0}, {0, 0, -1}, {0, 1, 0}};
Ly = {{0, 0, 1}, {0, 0, 0}, {-1, 0, 0}};
Lz = {{0, -1, 0}, {1, 0, 0}, {0, 0, 0}};
```

In[4]:=
```
MatrixForm[Lx.Ly - Ly.Lx ]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[5]:=
```
MatrixForm[Lx.Lz - Lz.Lx]
```

Out[5]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In[6]:=
```
MatrixForm[Ly.Lz - Lz.Ly]
```

Out[6]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Problem 7

We have to solve $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$ w/ $\psi(-a/2) = 0 = \psi(a/2)$

$$\psi(x) = A\cos\left(\sqrt{2mE}\,x/\hbar\right) + B\sin\left(\sqrt{2mE}\,x/\hbar\right)$$

$$\psi(-a/2) = A\cos\left(\sqrt{2mE}\,a/2\hbar\right) - B\sin\left(\sqrt{2mE}\,a/2\hbar\right) = 0$$

$$\psi(a/2) = A\cos\left(\sqrt{2mE}\,a/2\hbar\right) + B\sin\left(\sqrt{2mE}\,\frac{a}{2\hbar}\right) = 0$$

If we add these eqn.s, we get $2A\cos\left(\sqrt{2mE}\cdot a/2\hbar\right) = 0$

so

$$\sqrt{2mE}\,\frac{a}{2\hbar} = (2n+1)\pi/2 \quad \text{for } n = 0, 1, 2, \ldots$$

$$\to \quad E = \frac{(2n+1)^2\pi^2\hbar^2}{2ma^2}$$

Subtracting gives $\sin\left(\sqrt{2mE}\,\frac{a}{2\hbar}\right) = 0$ w/ $\sqrt{2mE}\,\frac{a}{2\hbar} = \underset{\uparrow\ \text{integer}}{n}\pi \Rightarrow E = \frac{(2n)^2\pi^2\hbar^2}{2ma^2}$

The cosine solution works for odd integers, the sine solution for even integers, so

$$\psi_j(x) \begin{cases} \sqrt{\frac{2}{a}}\cos\left(\frac{j\pi x}{a}\right) & j \text{ odd} \\ \sqrt{\frac{2}{a}}\sin\left(\frac{j\pi x}{a}\right) & j \text{ even} \end{cases}$$