Midterm Examination

Quantum Mechanics II
Physics 442

Due Date: March 8th, 2024

Instructions: There are four problems (ten points each). You may use any online/book resource you like (cite where appropriate) but may not work with other people on the exam, nor ask any real-time questions of the internet (i.e. no ChatGPT or similar). If you have any questions about what resources are available, or have any questions about what a problem is asking, please contact the course instructor. The examination is due by noon on Friday, March 8th (turn in via gradescope).

Problem 1

a. For a Hermitian matrix $\mathbb{A} \in \mathbb{C}^{n\times n}$, we define its square root to be a matrix $\mathbb{B}$ such that $\mathbb{B}\mathbb{B} = \mathbb{A}$. Assuming such a matrix exists, how would you construct it?

b. Show that the eigenvalues of an anti-Hermitian matrix, $\mathbb{A}^\dagger = -\mathbb{A}$, are imaginary. Assuming the eigenvalues are all distinct, how are the eigenvectors related (for Hermitian matrices, they are orthogonal)?

c. For two Hermitian matrices $\mathbb{A}$ and $\mathbb{B} \in \mathbb{C}^{n\times n}$, if $\mathbb{A}\mathbb{B} + \mathbb{B}\mathbb{A} = 0$, is it possible for the matrices to share one or more eigenvectors? If so, provide an example, if not, prove it.

Problem 2

Find the explicit form of the harmonic oscillator raising and lowering operators in the Heisenberg picture as functions of time, i.e. What are $\hat{a}_{H\,\pm}(t) =$?

Problem 3

The operator $\hat{Q}(a)$ acts on a wavefunction $\psi(x, y, z)$ (in position basis) according to:

$$\hat{Q}(a)\psi(x, y, z) = \psi(x + ay, y + ax, z).$$

a. Find the "infinitesimal generator" of this transformation, i.e. what operator, $\hat{J}$, has $(1 + (i\epsilon/\hbar)\hat{J})\psi(x, y, z)$ that matches $\hat{Q}(\epsilon)\psi(x, y, z)$ through order ϵ? From the infinitesimal generator, construct the full, unitary $\hat{Q}(a)$.

b. How do the operators $\hat{x}$, $\hat{y}$, $\hat{p}_x$ and $\hat{p}_y$ respond to the infinitesimal form of the transformation $\hat{Q}(\epsilon)$?

c. Evaluate $\frac{d\langle\hat{J}\rangle}{dt}$ for a free particle in three dimensions (you want $\frac{d\langle\hat{J}\rangle}{dt} = \Box$, where the right hand side should involve the expectation value of some function of the $\hat{x}$, $\hat{y}$, $\hat{p}_x$ and $\hat{p}_y$ operators).

Problem 4

For the potential energy

$$U(x) = \begin{cases} U_0 & 0 < x < a/2 \\ 0 & a/2 < x < a \\ \infty & x < 0 \text{ and } x > a. \end{cases}$$

Use the WKBJ approximation to find the energy spectrum.