Problem Set 5

Physics 442
Quantum Mechanics II

Due on Friday, March 1st, 2024

Problem 1

Given the single-particle states $|\ell_1 m_1\rangle = |\frac{1}{2}\frac{1}{2}\rangle$, $|\frac{1}{2}-\frac{1}{2}\rangle$ for "angular momentum" $\ell_1 = 1/2$ (a spin-one-half particle), the two-particle ($|\ell_1\ell_2 m_1 m_2\rangle$) states are

$$\left|\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right\rangle, \left|\frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}\right\rangle, \left|\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2}\right\rangle, \left|\frac{1}{2}\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\right\rangle. \tag{1}$$

The most general linear combination of these states is

$$|\Psi\rangle = \alpha\left|\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right\rangle + \beta\left|\frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}\right\rangle + \rho\left|\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2}\right\rangle + \sigma\left|\frac{1}{2}\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\right\rangle \tag{2}$$

Find the coefficients $\{\alpha, \beta, \rho, \sigma\}$ for the (normalized) states of total angular momentum: $|11\rangle$, $|10\rangle$ and $|1-1\rangle$ using the "two-angular-momentum" operators:

$$\hat{\mathbf{L}} = \hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2 \tag{3}$$

and any useful linear combinations of them. Construct the states, do not look them up (that's for the next problem).

Problem 2

Practice with Clebsch-Gordon Tables.

a. Using the table on p. 179, check your answers to Problem 1.

b. For the two-momenta states from the previous problem, find the coefficients that lead to $|00\rangle$ (again, using the table).

c. Combining a particle with spin $s_1 = 3/2$ with one that has spin $s_2 = 1/2$, write out the linear combination that has total-angular-momentum $S = 2$ with z-component $M = -1$ (using the table).

Problem 3

Griffiths & Schroeter Problem 6.21.

Problem 4

Griffiths & Schroeter Problem 6.24. Vector selection rules — you can "verify" rather than "derive" — take each of your commutator calculations from Part a., then use the relations in (6.59), (6.60), (6.61) to express your commutator calculation in terms of the A (coefficient associated with $\hat{V}_+$), B (coefficient associated with $\hat{V}_-$) and C (Clebsch-Gordon coefficients), and show that the resulting expression is contained in one of the recursion relations from (6.66).

Problem 5

Griffiths & Schroeter Problem 6.25.

Presentation Problems

These problems will either be presented in class on Friday (March 1st), or presented in written form, due by Friday. Take a look at them over the weekend, I'll ask for volunteers on Monday. Whether or not you present one of these problems or not, you should solve them and be prepared to discuss them in class.

Problem 1*

You showed that there is a complex number u that can be associated with points from the upper (unit) hemisphere (Problem Set 3, Problem 4). In order to allow for a mapping that includes the north pole (and the lower hemisphere, as well), take $u = p/q$ for complex p and q (and with $p = 1$, $q = 0$ associated with the north pole). Work out the relations between $\{p, q\}$ and $\{x, y, z\}$.

The numbers p and q can be embedded in a vector, defining the "spinor" $\chi = (p\ q)^T$. For a rotation of the Cartesian coordinates about the z axis: $\bar{\mathbf{x}} = \mathbb{R}_z \mathbf{x}$, what is the matrix $\mathbb{S}_z$ that appropriately rotates the spinner χ?

Problem 2*

One way to make classical mechanics "look" more like quantum mechanics is to think about the time-independent probability density associated with particle motion. In one dimension, for a particle traveling along with $x(t)$, we have $\rho(x,t) = \delta(x - x(t))$. For motion with period T, so that $x(t+T) = x(t)$, define the time average probability density:

$$\rho(x) \equiv \frac{1}{T/2} \int_0^{T/2} \rho(x,t) dt. \tag{4}$$

Write out the expression for $\rho(x)$ for an oscillatory particle of energy E moving under the influence of a potential energy $U(x)$.