Problem Set 3

Physics 442
Quantum Mechanics II

Due on Friday, February 16th, 2024

Problem 1

We have two Hermitian matrices $\mathbb{A}$ and $\mathbb{B}$, both in $\mathbb{C}^{n\times n}$, that commute with each other: $[\mathbb{A}, \mathbb{B}] = 0$. Assume the spectrum of each is nondegenerate, so that $\mathbb{A}\mathbf{v}^i = \lambda_i \mathbf{v}^i$ has no two λ_i alike, and similarly for $\mathbb{B}$. Show that the eigenvectors of $\mathbb{A}$ "are" (can be taken to be) shared by $\mathbb{B}$. The restrictions here can be relaxed, but the proof is easiest with them in place.

Problem 2

For the momentum operator p_j (with components $j = 1$, 2 and 3) and angular momentum operator L_i, what is the Poisson bracket of $[L_i, p_j]_{\mathsf{PB}} = \boxed{?}$[1] It is useful to write the angular momentum as $L_i = \sum_{\ell,k=1}^{3} \epsilon_{i\ell k} x_\ell p_k$ where $x_1 = x$, $x_2 = y$, $x_3 = z$ and similarly for the p_k. The "Levi-Civita" symbol is defined to be

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } 123 \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } 123 \\ 0 & \text{else.} \end{cases} \tag{1}$$

Using the prescription for turning Poisson Brackets into commutators, check that your expressions recover the $[\hat{L}_i, \hat{p}_j]$ commutator found on p. 250.

Problem 3

Given two matrices $\mathbb{A}$ and $\mathbb{B}$, both in $\mathbb{C}^{n\times n}$, show that if $\mathbb{A}$ and $\mathbb{B}$ do not commute, then

$$e^{\mathbb{A}+\mathbb{B}} \neq e^{\mathbb{A}} e^{\mathbb{B}}. \tag{2}$$

A counter-example will do.

[1] In this equation, both i and j run from 1 to 3, for a total of nine relations.

Problem 4

For a unit sphere centered at the origin, points in the northern hemisphere can be identified with points in the xy plane: For $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ pointing from the origin to a point in the northern hemisphere, make a vector that starts at the north pole and goes through $\mathbf{r}$ — find the location at which this vector strikes the xy plane, and by viewing the xy plane as the complex plane, write the strike point in the form $u = a + ib$ where a and b depend on x, y, z. Calculate the inverse transformation, where you find x, y and z as functions of u and its complex conjugate.

Problem 5

How do the spherical coordinates, r, θ and ϕ change under the coordinate inversion $\bar{x} = -x$, $\bar{y} = -y$, $\bar{z} = -z$ (i.e. what are $\bar{r}$, $\bar{\theta}$ and $\bar{\phi}$)?

Problem 6

Griffiths & Schroeter Problem 6.8.

Problem 7

Griffiths & Schroeter Problem 6.9.

Presentation Problems

These problems will either be presented in class on Friday (February 16th), or presented in written form, due by Friday. Take a look at them over the weekend, I'll ask for volunteers on Monday. Whether or not you present one of these problems or not, you should solve them and be prepared to discuss them in class.

Problem 1*

The "Laplace-Runge-Lenz" vector is defined by

$$\mathbf{R} \equiv \frac{1}{m}\mathbf{p} \times \mathbf{L} - \alpha\hat{\mathbf{r}}. \tag{3}$$

Show that all three of the quantities here have vanishing Poisson Bracket with the Hamiltonian:

$$H = \frac{p^2}{2m} - \frac{\alpha}{r}, \tag{4}$$

and identify the three infinitesimal transformations generated by $\mathbf{R}$.

Problem 2*

A "lab" (coordinates x, y, z and time t) L sits at rest, and another, $\bar{L}$ (coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$) travels with speed v along a shared $\hat{\mathbf{x}}$ axis (similar to the standard relativistic setup). Using non-relativistic expressions, find $\bar{x}$ as a function of x. If a particle of mass m had momentum $\bar{p}$ in $\bar{L}$, what would its momentum be in L? These expressions for $\bar{x}$ and $\bar{p}$ form a coordinate transformation with parameters v, t and m. Find the infinitesimal generating function, $J(x,p)$ that generates the transformation. Using this generator, and being careful with units, construct the quantum mechanical operator

$$\hat{G} = e^{\boxed{?}} \tag{5}$$

associated with this "Galilean transformation." What are the transformed $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ operators (use the infinitesimal form of $\hat{G}$ to work these out)?