Problem Set 2

Physics 442
Quantum Mechanics II

Due on Friday, February 9th, 2024

Problem 1

For a Hermitian matrix, $\mathbb{A} \in \mathbb{C}^{n\times n}$ with $\mathbb{A}^\dagger = \mathbb{A}$, define $\mathbb{U} = e^{i\mathbb{A}}$ using the series definition of the exponential. Hermitian matrices have real eigenvalues and eigenvectors that are ("can be taken to be," formally) orthonormal — we can write $\mathbb{A} = \mathbb{V}\mathbb{D}\mathbb{V}^\dagger$ where $\mathbb{D}$ is a diagonal matrix with the eigenvalues of $\mathbb{A}$ along the diagonal, and $\mathbb{V}$ is a unitary matrix: $\mathbb{V}^\dagger\mathbb{V} = \mathbb{I}$.

Write $\mathbb{U}$ in terms of $\mathbb{V}$, its conjugate transpose, and the matrix $e^{\mathbb{D}}$ which is the diagonal matrix with e^{λ_j}, for λ_j the eigenvalues of $\mathbb{A}$, as its entries.

Using this decomposition of $\mathbb{U}$, show that $\mathbb{U}^\dagger = e^{-i\mathbb{A}}$ and then that $\mathbb{U}^\dagger\mathbb{U} = \mathbb{I}$, to establish that $\mathbb{U}$ is unitary.

Problem 2

The infinitesimal form of the z-axis rotation generator is

$$\mathbb{L}_z \doteq \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Using the series definition of the exponential, write the entries of the matrix,

$$e^{\theta\mathbb{L}_z} = \boxed{?}, \tag{2}$$

where you compute the matrix exponential using techniques similar to the previous problem.

Problem 3

a. For $\Psi(\mathbf{r},t)$ satisfying the time-dependent Schrödinger's equation, show that, given a domain Ω with boundary $\partial\Omega$, the conservation statement:

$$\frac{d}{dt}\int_{\Omega}\Psi^*\Psi d\tau = -\oint_{\partial\Omega}\mathbf{J}\cdot d\mathbf{a} \tag{3}$$

holds by identifying the vector $\mathbf{J}$.

b. Using your result from part a., show that if you take Ω to be all space, you get

$$\frac{d}{dt}\int_{\text{all space}}\Psi^*\Psi d\tau = 0. \tag{4}$$

Problem 4

An alternate way to formulate the wavefunction $\Psi(\mathbf{r},t)$ is to use a polar representation of complex numbers:

$$\Psi(\mathbf{r},t) = R(\mathbf{r},t)e^{iS(\mathbf{r},t)} \tag{5}$$

for real functions $R(\mathbf{r},t)$, $S(\mathbf{r},t)$. Find the equations governing the amplitude $R(\mathbf{r},t)$ and phase, $S(\mathbf{r},t)$ – these form the basis for the Bohm interpretation of quantum mechanics. Express the ρ (probability density) and $\mathbf{J}$ (from the previous problem) in terms of the new R and S functions.

Problem 5

Using your results from problems 3 (with $J(x(t),p(t))$, but no explicit t-dependence) and 4 in problem set 1, write out the "quantum mechanical" relation between the time derivative of an *operator* J (we have yet to encounter such a thing, but it's coming). Take the expectation value of the resulting equation to get the usual form of Ehrenfest's theorem.

Problem 6

For the Hamiltonian with magnetic field in place,

$$H = \frac{1}{2m}(\mathbf{p}-q\mathbf{A})\cdot(\mathbf{p}-q\mathbf{A}), \tag{6}$$

use Ehrenfest's theorem to show that

$$\frac{d\langle \mathbf{r}\rangle}{dt} = \frac{1}{m}\langle \mathbf{p} - q\mathbf{A}\rangle. \tag{7}$$

Presentation Problems

These problems will either be presented in class on Friday (February 9th), or presented in written form, due by Friday. Take a look at them over the weekend, I'll ask for volunteers on Monday. Whether or not you present one of these problems or not, you should solve them and be prepared to discuss them in class.

Problem 1*

A Lorentz boost in the x-direction is described by the new-to-old coordinate relation:

$$\begin{pmatrix} c\bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{8}$$

For an infinitesimal boost, with $v \ll c$, write the matrix appearing on the right as $\mathbb{I} + \frac{v}{c}\mathbb{B}_x$ to isolate the infinitesimal form $\mathbb{B}_x$. Carry out a similar procedure to find $\mathbb{B}_y$ and $\mathbb{B}_z$. Compute all the commutators of these boost matrices.

Problem 2*

Suppose you have a solution to a magnetic-field problem in quantum mechanics,

$$\frac{1}{2m}\left[\mathbf{p} - q\mathbf{A}\right] \cdot \left[\mathbf{p} - q\mathbf{A}\right]\Psi = E\Psi. \tag{9}$$

where $\mathbf{p} \rightarrow \frac{\hbar}{i}\nabla$. Show that $\Phi \equiv e^{i\sigma}\Psi$ (for function of position σ) solves

$$\frac{1}{2m}\left[\mathbf{p} - q\mathbf{A} - q\nabla\varphi\right] \cdot \left[\mathbf{p} - q\mathbf{A} - q\nabla\varphi\right]\Phi = E\Phi, \tag{10}$$

given a function φ, provided $\sigma = \boxed{?}$. Does the relation between Φ and Ψ provide a compelling "gauge-invariance" story?

Problem 3*

Find the WKBJ-approximate energies and wavefunction for the potential:

$$U(x) = \begin{cases} \infty & x < 0 \text{ and } x > a \\ Px & 0 \leq x \leq a \end{cases} \tag{11}$$

for constant P.