Problem Set 10

Physics 442
Quantum Mechanics II

Due on Friday, April 19th, 2024

Problem 1

Griffiths & Schroeter Problem 10.21 — Neutron diffraction.

Problem 2

Griffiths & Schroeter Problem 11.2 — Hydrogen two-level transition.

Problem 3

Working in two dimensions, suppose we have, for E&M, an equation governing the electrostatic potential that looks like: $\nabla^2 V = -\rho/\epsilon$. Assuming that $\mathbf{F} = q\mathbf{E}$ as in three dimensions, what are the units of ϵ? Find the potential associated with a point charge q sitting at the origin. Sketch the effective one-dimensional potential (for $H = p^2/(2m) + qV(r)$ in two dimensions) associated with hydrogen. How much energy does it take to ionize two-dimensional hydrogen?

Problem 4

Suppose you have two solutions to Schrödinger's equation:

$$\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\psi_a + U\psi_a &= E_a\psi_a \\ -\frac{\hbar^2}{2m}\nabla^2\psi_b + U\psi_b &= E_b\psi_b. \end{aligned} \tag{1}$$

Show that for $E_a \neq E_b$, we have $\psi_a \perp \psi_b$ in the usual sense.

Problem 5

For a scattering state [1] associated with the step potential:

$$U(x) = \begin{cases} U_0 & x > 0 \\ 0 & x < 0 \end{cases} \tag{2}$$

compute $\rho = \Psi^*\Psi$ and $\mathbf{J} = i\hbar/(2m)(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)$. Apply the integral form of $\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J}$ to the closed domain, centered at the origin, shown below. From the resulting expression, identify the "reflection" and "transmission"

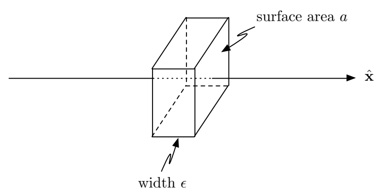

Figure 1: Domain for the conservation law evaluation in Problem 5.

coefficients, R and T, with $R + T = 1$. Note that you do not need to compute the relationship between the plane wave coefficients that come from continuity and derivative continuity.

Presentation Problems

These problems will either be presented in class on Friday (April 19th), or presented in written form, due by Friday. Take a look at them over the weekend, I'll ask for volunteers on Monday. Whether or not you present one of these problems or not, you should solve them and be prepared to discuss them in class.

Problem 1*

For the "Klein-Gordon" Lagrangian,

$$\mathcal{L} = -\frac{1}{c^2}\Psi_t\Psi_t^* + \Psi_x\Psi_x^* - \frac{iq}{\hbar c}U(\Psi\Psi_t^* - \Psi^*\Psi_t) + \frac{q^2}{h^2c^2}U^2\Psi^*\Psi + \left(\frac{mc}{\hbar}\right)^2\Psi^*\Psi, \tag{3}$$

[1] Use the usual convention: A plane wave comes in from the left, interacts with the potential, leading to a reflected wave on the left and a transmitted wave on the right.

find the field equation for Ψ. Introducing the usual infinite square well potential, U, for a well of width a, find the time-independent wave functions $\Psi(x)$ and energies E – use separation of variables, and assume the time-dependent solution is of the form: $T(t) = e^{-iEt/\hbar}$ "as usual."

Problem 2*

The Crank-Nicolson method for approximating time-dependent solutions to the Schrödinger equation has update given by (see Presentation Problem 1 from Problem Set 9 for details):

$$i\hbar\left(\frac{\boldsymbol{\psi}^{n+1} - \boldsymbol{\psi}^{n}}{\Delta t}\right) = \mathbb{H}\left(\frac{\boldsymbol{\psi}^{n+1} + \boldsymbol{\psi}^{n}}{2}\right). \tag{4}$$

Implement the method, and use it with zero boundary conditions at $x = 0$ and $x = a$ to examine the time-evolution of an initial Gaussian in an infinite square well (no potential) — center your Gaussian, and make sure that it is sharp enough that its "tails" are close to zero initially. Try giving your initial wavefunction an initial momentum to the right and see what happens.

Problem 3*

Griffiths & Schroeter Problem 11.9 — Rabi flopping frequency.