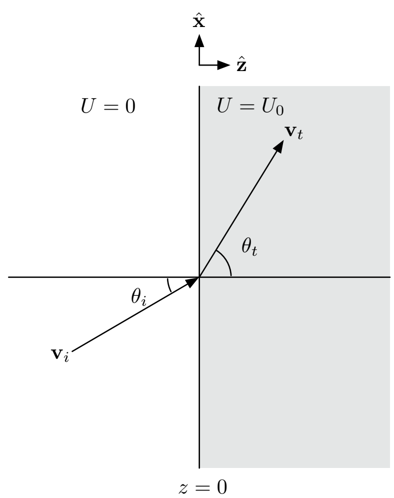
Problem Set 1

Physics 442
Quantum Mechanics II

Due on Friday, February 2nd, 2024

Problem 1

A (classical) particle of mass m travels from $z < 0$ into a region of non-zero potential energy U_0 with $z > 0$. If the "angle of incidence" is θ_i (associated with the incoming velocity vector $\mathbf{v}_i = v \sin\theta_i \hat{\mathbf{x}} + v\cos\theta_i \hat{\mathbf{z}}$), what is the transmitted angle θ_t associated with the "outgoing" velocity vector $\mathbf{v}_t$? The setup, with angles shown, is shown below:



Problem 2

For a Lagrangian written in spherical coordinates,

$$L = \frac{1}{2}m\Big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\Big) - U(r,\theta,\phi) \tag{1}$$

develop the associated Hamiltonian. If the potential energy U does not depend on ϕ, show that there is a constant of the motion and give its form in terms of r, θ, ϕ and their time derivatives.

Problem 3

The Poisson bracket of functions $f(x,p)$ and $g(x,p)$ is defined to be

$$\{f,g\} \equiv \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}. \tag{2}$$

Evaluate the Poisson bracket of x and p: $\{x,p\} = \boxed{?}$. How would you turn this result into a quantum mechanical commutator, i.e. what is the rule taking $\{x,p\}$ to the quantum $[x,p]$?

Problem 4

For a function $J(x(t),p(t),t)$, write the total time derivative of J evaluated along a Hamiltonian trajectory (meaning that $x(t)$ and $p(t)$ satisfy the Hamilton equations of motion). Express your solution in terms of the "Poisson Bracket" defined in (2).

Problem 5

A particle of mass m with charge q moves in a magnetic field with vector potential $\mathbf{A}$, a function of position (but, you may assume, not time). The Hamiltonian governing its motion is

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})\cdot(\mathbf{p} - q\mathbf{A}). \tag{3}$$

Find the equations of motion for this Hamiltonian, and show that they are equivalent to $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}$.

Problem 6

For a rotation about the $\hat{\mathbf{z}}$ axis, we have new coordinates:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\equiv \mathbb{O}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{4}$$

Write the infinitesimal form (for small θ) of the matrix $\mathbb{O}$ appearing in the transformation, it should take the form: $\mathbb{O} \approx \mathbb{I} + \theta \mathbb{L}_z$. By considering infinitesimal rotations about the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ axes (keeping the sense of the rotation the same), identify the set $\mathbb{L}_x$, $\mathbb{L}_y$ and $\mathbb{L}_z$, and compute the commutators of these matrices. Do they look familiar?

Problem 7

Find the bound state wavefunctions and energies for the "centered" square well with potential energy:

$$U(x) = \begin{cases} \infty & x < -a/2 \text{ and } x > a/2 \\ 0 & -a/2 < x < a/2 \end{cases} \tag{5}$$