# Born Approximation



For $\Psi = A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right]$

$e^{ikz}$ ← incoming, $f(\theta)\frac{e^{ikr}}{r}$ ← outgoing

w/ $d\sigma = |f(\theta)|^2 d\Omega$

$|f(\theta)|^2 = D(\theta)$ the diff. x-sec.

We had:

$$\Psi(\vec{r}) = \underbrace{\Psi_0(\vec{r})}_{= e^{ikz}} + \frac{2m}{\hbar^2}\int_{\text{all space}} G(\vec{r},\vec{r}\,')\,U(\vec{r}\,')\,\Psi(\vec{r}\,')\,d\tau' \quad (*)$$

for Green's function $G(\vec{r},\vec{r}\,') = -\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}$

and assuming a localized potential, we can find $\Psi$ for $r \gg r'$:

$|\vec{r}-\vec{r}\,'| \approx r\left(1 - \frac{\hat{r}\cdot\vec{r}\,'}{r}\right) = r - \hat{r}\cdot\vec{r}\,'$, w/

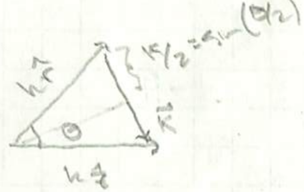
$G(\vec{r},\vec{r}\,') \approx -\frac{e^{ikr}\cdot e^{-ik\hat{r}\cdot\vec{r}\,'}}{4\pi r}$ - putting this in $(*)$:

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \frac{e^{ikr}}{r}\underbrace{\left[-\frac{m}{2\pi\hbar^2}\int_{\text{all space}} e^{-ik\hat{r}\cdot\vec{r}\,'}\,U(\vec{r}\,')\,\Psi(\vec{r}\,')\,d\tau'\right]}_{=f(\theta)}$$

we approximate $\Psi(\vec{r}\,')$ in the int. w/ $\Psi_0(\vec{r}\,') = e^{ikz'}$

$$f(\theta) \approx -\frac{m}{2\pi\hbar^2}\int_{\text{all space}} e^{ik(\hat{z}-\hat{r})\cdot\vec{r}\,'}\,U(\vec{r}\,')\,d\tau'$$

we defined $\vec{K} = k\hat{z} - k\hat{r}$



w/

$K = 2k\sin(\theta/2)$

$\rightarrow$

$k^2 = \frac{2mE}{\hbar^2}$.

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int_{\text{all space}} e^{iKr'\cos\theta'}\,U(\vec{r}\,')\,d\tau'$$

If $U$ is spherically symmetric: $U(\vec{r}\,') = U(r')$

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int_0^\infty\int_0^\pi\int_0^{2\pi} e^{iKr'\cos\theta'}\,U(r')\,r'^2\sin\theta'\,d\phi'\,d\theta'\,dr'$$

$$= -\frac{m}{\hbar^2}\int_0^\infty\int_0^\pi \frac{d}{d\theta'}\left[\frac{-e^{iKr'\cos\theta'}}{iKr'}\right]U(r')\,d\theta'\,dr'$$

$$= +\frac{m}{\hbar^2}\int_0^\infty \frac{1}{iK}\underbrace{\left[e^{-iKr'} - e^{iKr'}\right]}_{=-2i\sin(Kr')} r'\,U(r')\,dr'$$

$$= -\frac{2m}{\hbar^2 K}\int_0^\infty r'\,U(r')\sin(Kr')\,dr'$$

Example - Coulomb

For $U(r) = \frac{q^2}{4\pi\epsilon_0 r}$, we can find $f(\theta)$ - issue:

$$f(\theta) = -\frac{2mq^2}{4\pi\epsilon_0\hbar^2 K}\int_0^\infty \sin(Kr')\,dr' = ?$$

Introduce a cut-off: $U(r) = \frac{q^2 e^{-\mu r}}{4\pi\epsilon_0 r}$ & send $\mu \to 0$ at the end.

$$f(\theta) = -\frac{mq^2}{2\pi\epsilon_0 \hbar^2 K}\int_0^\infty e^{-\mu r'}\sin(Kr')\,dr'$$

& the integral is:

$$\int_0^\infty e^{-\mu r'}\sin(Kr')\,dr' = \frac{1}{2i}\int_0^\infty \left[e^{-\mu r' + iKr'} - e^{-\mu r' - iKr'}\right]dr'$$

$$= \frac{1}{2i}\left[\frac{e^{-\mu r'}e^{iKr'}}{-\mu + iK}\Big|_0^\infty + \frac{e^{-\mu r'}e^{-iKr'}}{\mu + iK}\Big|_0^\infty\right]$$

$$= \frac{1}{2i}\left[\frac{-1}{-\mu + iK} - \frac{1}{\mu + iK}\right] = \frac{-1}{2i}\left[\frac{\mu + iK - \mu + iK}{-\mu^2 - K^2}\right]$$

$$= \frac{K}{\mu^2 + K^2}$$

so $$f(\theta) = -\frac{mq^2}{2\pi\epsilon_0 \hbar^2}\frac{1}{(\mu^2 + K^2)}$$

& set $\mu \to 0$ to get the Coulomb version:

$$f(\theta) = -\frac{mq^2}{2\pi\epsilon_0 \hbar^2}\frac{1}{\left(4\cdot\frac{2mE}{\hbar^2}\sin^2(\theta/2)\right)} = \frac{-q^2 m}{16\pi\epsilon_0 \sin^2(\theta/2)}$$

& $$D(\theta) = |f(\theta)|^2 = \left(\frac{mq^2}{16\pi\epsilon_0}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

no $\hbar^2$ ... identical to the classical result.

## Time Dependence

For a time-indep. $\hat{H}$, suppose we have $|\psi_j\rangle$ & $E_j$ w/

$$\hat{H}|\psi_j\rangle = E_j|\psi_j\rangle$$

then: $|\psi(t)\rangle = \sum_{j=1}^\infty c_j e^{-iE_j t/\hbar}|\psi_j\rangle$ w/ the $c_j$ determined by the initial condition:

$$|\psi(0)\rangle = \sum_{j=1}^\infty c_j |\psi_j\rangle = |\psi_0\rangle$$

so

$$\langle\psi_k|\psi_0\rangle = c_k \checkmark$$

the $E_j$ are the "allowed energies" for measurement.

? Suppose you measure the energy $E_\ell$ – what happens after that measurement?

$$|\psi\rangle = ?$$

Take as our system an infinite sq. well, w/

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \quad \text{in pos. rep.}$$

the $|\psi(t)\rangle$ above satisfies:

$$\hat{H}_0|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle$$

? What happens if we introduce a time-varying potential $U(t)$?

suppose we have $\hat{H}' = U(x,t)$, then we can start w/

$$|\Psi(t)\rangle = \sum_{j=1}^{\infty} c_j(t) e^{-iE_jt/\hbar} |\psi_j\rangle$$

Running this through: $(\hat{H}_0 + \hat{H}')|\Psi(t)\rangle = i\hbar \frac{d}{dt}|\Psi(t)\rangle$ gives:

$$(\hat{H}_0 + \hat{H}')|\Psi(t)\rangle = \sum_{j=1}^{\infty} c_j(t) e^{-iE_jt/\hbar} [E_j + U]|\psi_j\rangle$$

on the left, w/

$$i\hbar \frac{d}{dt}|\psi_j\rangle = \sum_{j=1}^{\infty} (i\hbar \dot{c}_j(t) + E_j c_j(t)) e^{-iE_jt/\hbar} |\psi_j\rangle$$

putting the 2 sides together & hitting them w/ $\langle\psi_k|$ gives:

same

$$[i\hbar \dot{c}_k(t) + E_k c_k(t)] e^{-iE_kt/\hbar} = c_k(t) e^{-iE_kt/\hbar} E_k + \sum_{j=1}^{\infty} c_j(t) e^{-iE_jt/\hbar} \underbrace{\langle\psi_k|U|\psi_j\rangle}_{\equiv H'_{kj}}$$

so

$$i\hbar \dot{c}_k(t) = \sum_{j=1}^{\infty} c_j(t) e^{-i(E_j - E_k)t/\hbar} H'_{kj}$$

or $$\dot{c}_k(t) = \frac{1}{i\hbar} \sum_{j=1}^{\infty} c_j(t) e^{-i(E_j - E_k)t/\hbar} H'_{kj}$$

For a "2-level system" w/ states $|\psi_a\rangle$ & $|\psi_b\rangle$

$(\hat{H}_0|\psi_a\rangle = E_a|\psi_a\rangle$, $\hat{H}_0|\psi_b\rangle = E_b|\psi_b\rangle$ w/ $\langle\psi_a|\psi_b\rangle = 0)$

$$\dot{c}_a(t) = \frac{1}{i\hbar}\left[c_a(t) H'_{aa} + c_b(t) e^{-i(E_b - E_a)t/\hbar} H'_{ab}\right]$$

let $\omega_0 \equiv \frac{E_b - E_a}{\hbar}$, then

$$\dot{c}_a(t) = \frac{-i}{\hbar}\left[c_a(t) H'_{aa} + c_b(t) e^{-i\omega_0 t} H'_{ab}\right]$$

$$\dot{c}_b(t) = \frac{-i}{\hbar}\left[c_b(t) H'_{bb} + c_a(t) e^{i\omega_0 t} H'_{ba}\right]$$

or

$$\frac{d}{dt}\begin{pmatrix} c_a(t) \\ c_b(t) \end{pmatrix} = \frac{-i}{\hbar}\begin{pmatrix} H'_{aa} & H'_{ab} e^{-i\omega_0 t} \\ H'_{ba} e^{i\omega_0 t} & H'_{bb} \end{pmatrix}\begin{pmatrix} c_a(t) \\ c_b(t) \end{pmatrix}$$

note that: $(H'_{ab} e^{-i\omega_0 t})^* = H'_{ba} e^{i\omega_0 t}$, the matrix here is Hermitian.

This would be relatively easy to solve if the $H'$ matrix elt.s were t-independent...