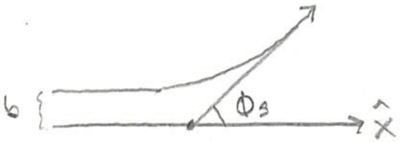
Coulomb Cross Section



We need to solve:

$$\rho''(\phi) = -\rho(\phi) - \frac{m}{L_z^2}\frac{dU}{d\rho}. \qquad (*)$$

w/ $\rho(\pi) = 0$, $\rho'(\pi)^2 = \frac{2mE_0}{L_z^2} = \frac{1}{b^2}$ ← which root should we take?

w/ $E_0 = \frac{1}{2} m v_0^2$ & $L_z^2 = 2mb^2E_0$.

For the Coulomb potential: $U = +\frac{kq^2}{r} = +kq^2\rho$.
(& $k = 1/4\pi\epsilon_0$)

Then (*) becomes:

$$\rho''(\phi) = -\rho(\phi) - \frac{kq^2m}{L_z^2}$$

The solution is: $\rho(\phi) = A\cos\phi + B\sin\phi - \frac{kq^2m}{L_z^2}$.

$\rho(\pi) = -A - \frac{kq^2m}{L_z^2} = 0$ gives A, &

$\rho'(\pi) = -B = -\frac{1}{b}$

so

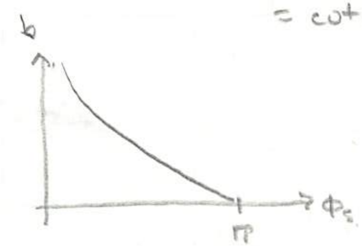
$$\rho(\phi) = -\frac{kq^2m}{L_z^2}(1+\cos\phi) + \frac{1}{b}\sin\phi$$
$$\stackrel{!}{=} -\frac{kq^2}{2b^2E_0}(1+\cos\phi) + \frac{1}{b}\sin\phi$$

the scattering angle Φ_s is defined by:

$$\rho(\Phi_s) = 0$$

and we can solve this eqn. for b:

$$b = \frac{kq^2}{2E_0}\underbrace{\frac{(1+\cos\Phi_s)}{\sin\Phi_s}}_{=\cot(\Phi_s/2)} = \frac{kq^2}{2E_0}\cot(\Phi_s/2).$$



- undeflected path, $\Phi_s = 0$, occurs only for $b \to \infty$.
- $\Phi_s = \pi$ is head on collision

Putting this b in to the usual scattering format

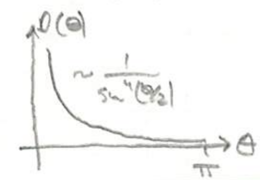
$$b(\theta) = \frac{q^2}{8\pi\epsilon_0 E_0}\cot(\theta/2)$$

the differential scattering cross section is

$$D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \quad \text{w/} \quad \frac{d}{d\theta}\left(\frac{\cos(\theta/2)}{\sin(\theta/2)}\right) = -\frac{1}{2}\left[1 + \frac{\cos^2(\theta/2)}{\sin^2(\theta/2)}\right] = \frac{-1}{2\sin^2(\theta/2)}$$

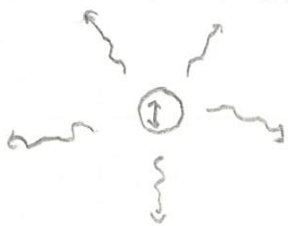
$$= \left(\frac{q^2}{8\pi\epsilon_0 E_0}\right)^2 \frac{\cot(\theta/2)}{2\sin\theta\,\sin^2(\theta/2)} \qquad \frac{\cot(\theta/2)}{2\sin\theta\,\sin^2(\theta/2)} = \frac{1}{2\sin^4(\theta/2)}$$

$$= \left(\frac{q^2}{16\pi\epsilon_0 E_0}\right)^2 \frac{1}{\sin^4(\theta/2)}$$



Scattering of Fields

The "stuff" starts oscillating & radiating.



The incoming plane wave has the form:

$$\vec{E}_{in} = E_0 \underbrace{e^{i(kz-\omega t)}}_{\text{travels in z-dir.}} \hat{x} \leftarrow \text{polarized in the x-dir.}$$

The outgoing wave has the form:

$$\vec{E}_{out} = \frac{K e^{i(kr-\omega t)}}{r} \sin\theta\, \hat{\theta}$$

Focusing on the spatial dependence:

$$E_{in} = E_0 e^{ikz} \qquad E_{out} = \frac{e^{ikr}}{r} F(\theta)$$

The power/area transported by the waves is given by the Poynting vector:

$$\vec{I}_{in} \equiv \langle \vec{S}_{in} \rangle = \left\langle \frac{1}{\mu_0} \vec{E}_{in} \times \vec{B}_{in} \right\rangle \sim \frac{\frac{1}{2} E_0^2}{\mu_0 c} \hat{z}$$

intensity (time avg. of $\vec{S}$)

and the outgoing power/area is:

$$\vec{I}_{out} \equiv \langle \vec{S}_{out} \rangle = \left\langle \frac{1}{\mu_0} \vec{E}_{out} \times \vec{B}_{out} \right\rangle \sim \frac{F(\theta)^2}{r^2} \hat{r}$$

the "particle scattering setup" now looks like:

& energy conservation says: $S_{in}\, d\sigma = S_{out}\, r^2 d\Omega$

$$d\sigma = \underbrace{\frac{S_{out}}{S_{in}} r^2}_{\equiv D(\theta)} d\Omega \quad \text{differential scattering cross section.}$$

For E&M plane waves: $D(\theta) = F(\theta)^2 = \underbrace{\quad}_{\text{const.}} \sin^2\theta$.

Quantum Mechanical Scattering

In QM, scattering looks more like E&M. our incident beam is described by a free particle travelling in the z-direction:

$$\psi_{in} = \psi_0 e^{ikz}$$

the outgoing free particle is:

$$\psi_{out} = f(\theta) \frac{e^{ikr}}{r} \quad \text{(a spherical wave function)}$$

conservation of probability now gives: $|\psi_{in}|^2 (v\,dt)\, d\sigma = |\psi_{out}|^2 (v\,dt)\, r^2 d\Omega$

so $d\sigma = \dfrac{|f(\theta)|^2}{|\psi_0|^2} d\Omega$ so $D(\theta) = |f(\theta)|^2$.

Integral Solution to Schrödinger's Equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = E\psi$$

can be written as:

$$\nabla^2\psi + k^2\psi = s$$

$k^2 = \frac{2mE}{\hbar^2}$, $s = \frac{2mU}{\hbar^2}\psi$, a func. of pos. ...

If we had the "Green's function" for this eqn., $G(\vec{r},\vec{r}')$ solving

$$\nabla^2 G(\vec{r},\vec{r}') + k^2 G(\vec{r},\vec{r}') = \delta^3(\vec{r}-\vec{r}') \quad (*)$$

then

$$\psi(\vec{r}) = \int_{\text{all space}} G(\vec{r},\vec{r}')\, s(\vec{r}')\, d\tau'$$

since:

$$\nabla^2\psi + k^2\psi = \int_{\text{all space}} \delta^3(\vec{r}-\vec{r}')\, s(\vec{r}')\, d\tau' = s(\vec{r}) \checkmark$$

=

To solve $(*)$, set $\vec{r}'=0$ & assume $G(\vec{r},\vec{r}') = G(r)$ (spherical symmetry), & solve at points away from the origin:

$$\nabla^2 G(r) + k^2 G(r) = 0$$

$$\frac{1}{r}(rG)'' + k^2 G = 0 \Rightarrow (rG)'' = -k^2(rG),$$

so that $rG = Ae^{\pm ikr} \Rightarrow G = \frac{Ae^{\pm ikr}}{r}$

(motivated the form of ψ_{out})

To get A, note that when $k=0$, $G = A/r$, & integrating $(*)$ over a ball of radius ϵ:

$$\int_{B(0,\epsilon)} \nabla^2 G = \int_{B(0,\epsilon)} \delta^3(\vec{r})\, d\tau = 1$$

div. thm. "

$$\oint_{\partial B(0,\epsilon)} \nabla G \cdot d\vec{a} = \frac{-A}{\epsilon^2}\cdot \epsilon^2 \cdot 4\pi = 1 \Rightarrow A = -\frac{1}{4\pi}. \text{ (holds } \forall k)$$

& moving the source back to $\vec{r}'$ gives

$$G(\vec{r},\vec{r}') = -\frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}$$