Klein-Gordon

For the one-dimensional Hamiltonian:

$$E = mc^2\sqrt{1+\left(\frac{p}{mc}\right)^2} + U(x)$$

we take: $(E-U(x))^2 = m^2c^4 + p^2c^2$

& let $p \longrightarrow \frac{\hbar}{i}\frac{d}{dx}$ to get the time-independent Klein-Gordon eqn: ✓

$$(E-U(x))^2\,\Phi(x) = m^2c^4\Phi(x) - c^2\hbar^2\frac{d^2\Phi(x)}{dx^2}$$

or

$$\Phi''(x) = -\frac{1}{\hbar^2c^2}\left[(E-U(x))^2 - m^2c^4\right]\Phi$$

The conserved current $J^\mu = i[\Phi D^{\mu *}\Phi^* - \Phi^* D^\mu\Phi]$

w/ $D^\mu \doteq \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} - \frac{iq}{\hbar}V/c \\ \nabla \end{pmatrix}$

so that $J^0 = i\left[\Phi\left(-\frac{1}{c}\dot\Phi^* + \frac{iq}{\hbar c}V\Phi^*\right) - \left(-\frac{1}{c}\dot\Phi - \frac{iq}{\hbar c}V\right)\Phi^*\right]$

$$\stackrel{!}{=} i\left[-\frac{1}{c}\left(\Phi\dot\Phi^* - \dot\Phi\Phi^*\right) + 2\frac{iq}{\hbar c}V\Phi^*\Phi\right]$$

$$\stackrel{!}{=} \frac{i}{c}\left[-\Phi\dot\Phi^* + \dot\Phi\Phi^* + 2\frac{iqV}{\hbar}\Phi^*\Phi\right]$$

in this time-independent setting, $-\frac{\hbar}{i}\frac{\partial}{\partial t} \longrightarrow E \Rightarrow i\frac{\partial}{\partial t} \longrightarrow E/\hbar$,

so that

$$J^0 = \frac{1}{\hbar c}\left[+E\Phi\Phi^* + E\Phi\Phi^*\right] - 2\frac{qV}{\hbar c}\Phi^*\Phi$$

$$\stackrel{!}{=} \frac{2}{\hbar c}\underbrace{(E-qV)}_{=U(x)\text{ the potential energy}}\Phi^*\Phi.$$

we have already lost our probabilistic interpretation for $\Phi^*\Phi$ - prob. interp. requires conservation

For Sch.: $\frac{\partial}{\partial t}(\psi^*\psi) = -\nabla\cdot\vec{J}$

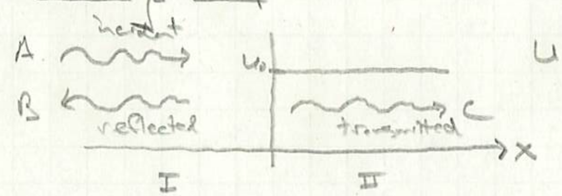
but on the KG side: $\frac{\partial}{\partial t}\left(\frac{1}{c}J^0\right) = -\nabla\cdot\vec{J}$

& $J^0 \not\propto \Phi^*\Phi$.

We still have $\vec{J} = i(\Phi\nabla\Phi^* - \Phi^*\nabla\Phi)$.

J^0 is still some sort of conserved density - note that it can be + or -, so what is conserved here can be pos. or neg. ... charge! ✓

Scattering Setup



$$U(x) = \begin{cases} 0 & x<0 \\ U_0 & x>0 \end{cases}$$

solve, use continuity & derivative continuity.

I: $\Phi_I'' = -\underbrace{\frac{1}{\hbar^2c^2}(E^2-m^2c^4)}_{\equiv k^2 \geq 0}\Phi_I$ since $E > mc^2$

then $\Phi_I = Ae^{ikx} + Be^{-ikx}$

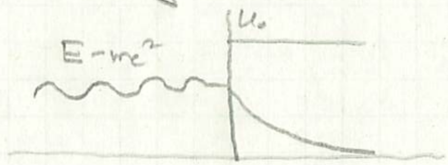
Very similar to the Schrödinger case.

in region II: $\Phi_{II}'' = -\underbrace{\frac{1}{\hbar^2 c^2}\left[(E-U_0)^2 - m^2c^4\right]}_{\equiv K^2}\Phi_{II}$ (*)

there are 2 options:

1. $K^2 < 0$ for $(E-U_0) < mc^2 \Rightarrow (E-mc^2) < U_0$

this gives a tunneling scenario:



that's not scattering, though, so moving on to:

2. $K^2 > 0$ - we get oscillatory solutions for Φ_{II}.

this can happen in 2 ways: $(E-U_0)^2 - m^2c^4 \geq 0$

a. $E - U_0 \geq mc^2$

b. $E - U_0 \leq -mc^2$

the charge density, $J^0 = \frac{2}{\hbar c}(E-U_0)\Phi^*\Phi$ on the right.

In case a., we have $J^0 > 0$, positive charge.

For case b., $J^0 < 0$, negative charge.

the solution to (*) is:

$$\Phi_{II} = Ge^{iKx} + He^{-iKx}$$

the e^{iKx} piece has $i(\Phi_{II}\nabla\Phi_{II}^* - \Phi_{II}^*\nabla\Phi_{II}) \sim K\hat{x}$

& e^{-iKx} has $\vec{J} \sim -K\hat{x}$

Now if $J^0 > 0$, we associate $\vec{J} \sim K\hat{x}$ w/ positive charge moving to the right ($J \sim \rho v$), we do not allow, in the scattering setup, particles moving to the left in II.

$J^0 > 0$ $\quad \Phi_{II} = Ge^{iKx}$

If $J^0 < 0$, we'll take the e^{-iKx} w/ $\vec{J} \sim -K\hat{x}$, but we understand this as negative charge moving right.

Combining the two solutions:

$$\Phi_{II}(x) = Ge^{\pm iKx} \quad \text{w/ } + \text{ for } J^0 > 0, \; - \text{ for } J^0 < 0.$$

Now we find the relations between A, B & G using continuity & derivative continuity.

Take the $+iKx$ sign to make things easier:

$$\Phi_I(0) = \Phi_{II}(0) \Rightarrow A + B = G$$

$$\Phi_I'(0) = \Phi_{II}'(0) \Rightarrow ik(A-B) = iKG$$

so $A+B=G$, $A-B=\frac{K}{k}G$ $\xrightarrow[\text{sub.}]{\text{add}}$ $A=\frac{1}{2}\left(1+\frac{K}{k}\right)G$

$B=\frac{1}{2}\left(1-\frac{K}{k}\right)G$

so $G=\dfrac{A}{\frac{1}{2}(1+K/k)}$ $\quad B=\dfrac{(1-K/k)}{(1+K/k)}A.$

applying conservation at the step, you showed in the $\sqrt{}$Sch. case!

$$1=\underbrace{\frac{|B|^2}{|A|^2}}_{R}+\underbrace{\frac{K}{k}\frac{|G|^2}{|A|^2}}_{T}$$

& the same holds here:

$$R=\frac{(1-K/k)^2}{(1+K/k)^2}=\frac{(k-K)^2}{(k+K)^2}$$

$$T=\frac{4K/k}{\frac{1}{k^2}(k+K)^2}=\frac{4Kk}{(k+K)^2}$$

Note that $R+T=\dfrac{k^2-2kK+K^2+4kK}{(k+K)^2}=1$ ✓

But now, we can have $\pm K$, giving

$$R=\frac{(k\mp K)^2}{(k\pm K)^2} \quad \& \quad T=\frac{\pm 4Kk}{(k+K)^2}$$

but now we see that we also need to think about

$R>1$ & $T<0$ (still w/ $R+T=1$, though).

negative charges moving right in II does the trick — you get equal & opposite positive charge moving left in I, hence $R>1$.

(some "reflected," some "produced"