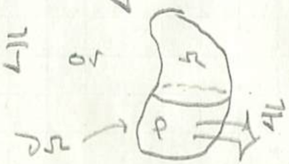
Conserved Current

In QM, there is a conservation law associated w/ the Schrödinger eqn:

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot\vec{J} \quad \text{or}$$



$$\frac{d}{dt}\int_{\Omega} \rho\, d\tau = -\oint_{\partial\Omega} \vec{J}\cdot d\vec{\sigma}$$

w/ ρ the "probability density" $\rho = \psi^*\psi$

& $$\vec{J} = \frac{i\hbar}{2m}\left[\psi\nabla\psi^* - \psi^*\nabla\psi\right]$$

For the Klein-Gordon eqn., there is also a conservation law, a "J^μ" w/

$$\partial_\mu J^\mu = 0,\quad J^\mu \doteq \begin{pmatrix} J^0 \\ \vec{J} \end{pmatrix},\quad \partial_\mu \doteq \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \nabla \end{pmatrix}$$

$$\Downarrow$$

$$\frac{1}{c}\frac{\partial J^0}{\partial t} + \nabla\cdot\vec{J} = 0$$

the J^μ is a 4-vector w/:

$$J^\mu \propto i\left[\psi(D^\mu\psi)^* - \psi^* D^\mu \psi\right] \quad (*)$$

In the absence of charge,

$$D^\mu = \partial^\mu \doteq \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} \\ \nabla \end{pmatrix}$$

but we know that when, for example, magnetism is involved, the classical H is:

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})\cdot(\vec{p} - q\vec{A})$$

& then $\vec{p} - q\vec{A} \longrightarrow \frac{\hbar}{i}\nabla - q\vec{A}$,

write this as:

$$\vec{D} \equiv \nabla - \frac{iq}{\hbar}\vec{A}$$

looks like the spatial part of

$$D^\mu = \partial^\mu - \frac{iq}{\hbar}A^\mu \quad \text{w/ } A^\mu \doteq \begin{pmatrix} V/c \\ \vec{A} \end{pmatrix}$$

this is the D^μ appearing in $(*)$.

Suppose you have an electrostatic configuration, $\vec{A} = 0$ & some V.

Then $$D^\mu \doteq \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} - \frac{iq}{\hbar c}V \\ \nabla \end{pmatrix}$$

the J^0 from $(*)$ is:

$$J^0 \doteq i\left[\psi\left(-\frac{1}{c}\frac{\partial}{\partial t} + \frac{iq}{\hbar c}V\right)\psi^* - \psi^*\left(-\frac{1}{c}\frac{\partial}{\partial t} - \frac{iq}{\hbar c}V\psi\right)\right]$$

$$\doteq \frac{i}{c}\left[-\psi\dot{\psi}^* + \psi^*\dot{\psi} + \frac{2iq}{\hbar}V\psi^*\psi\right]$$

& $$\vec{J} = i\left[\psi\nabla\psi^* - \psi^*\nabla\psi\right].$$