## Harmonic Oscillator "Lifetime"

We've been imaging hydrogen as our model 2-level system, but we can be concrete & work explicitly w/ the oscillator.

The raising & lowering operators are:

$$\hat{a}_\pm = \frac{1}{\sqrt{2m\hbar\omega}}\left(\mp i\hat{p} + m\omega\hat{x}\right), \quad [a_-, a_+] = 1$$

w/

$$\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle, \quad \hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle$$

The spectrum is: $E_n = (n+\tfrac{1}{2})\hbar\omega \quad n = 0, 1, \ldots$

Let's compute the "A" coefficient for spontaneous emission in this setting:

$$A = \frac{|p|^2 \omega_0^3}{3\pi\epsilon_0\hbar c^3} \leftarrow \text{freq. associated w/ the transition.}$$

w/ $p = \langle n'|q\hat{x}|n\rangle = q\langle n'|\hat{x}|n\rangle$

From the def. of $\hat{a}_\pm$, we can invert to get

$$\hat{x} = \underbrace{\left(\frac{\hbar}{2m\omega}\right)^{1/2}}_{\equiv K}(\hat{a}_+ + \hat{a}_-)$$

so:

$$\langle n'|\hat{x}|n\rangle = K\left[\langle n'|\hat{a}_+|n\rangle + \langle n'|\hat{a}_-|n\rangle\right]$$

$$= K\left[\sqrt{n+1}\,\langle n'|n+1\rangle + \sqrt{n}\,\langle n'|n-1\rangle\right]$$

the only allowed transitions are between $|n\rangle$ & $|n\pm 1\rangle$

for spontaneous emission, we want $|n'\rangle = |n-1\rangle$

$$p = \langle n-1|q\hat{x}|n\rangle = q\left(\frac{\hbar}{2m\omega}\right)^{1/2}\sqrt{n}, \quad \& \quad \omega_0 = \omega \quad \left(= \frac{E_n - E_{n-1}}{\hbar}\right)$$

& then

$$A = \frac{nq^2\omega^2}{6\pi\epsilon_0 m c^3}, \text{ the spontaneous emission rate.}$$

the lifetime of the state is $\tau = 1/A$.

The power lost to the oscillator is

$$P = \underset{\text{rate}}{A} \cdot \hbar\omega \leftarrow \text{energy loss} = \frac{q^2\omega^2}{6\pi\epsilon_0 m c^3} \cdot \underbrace{n\hbar\omega}_{= E_n - \frac{1}{2}\hbar\omega}$$

$$= \underbrace{\frac{q^2\omega^2}{6\pi\epsilon_0 m c^3}}_{\text{no } \hbar \text{ - matches classical result.}}\left(E - \tfrac{1}{2}\hbar\omega\right)$$

## Fermi's Golden Rule

Our perturbation work has been on a 2 level system between bound states.

We can also work out transitions between bound states & scattering states.

First order transition probability for 2-level system:

$$P(t) = \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}$$

& we'll think of this as a transition from a to b

Now $|b\rangle$ is a scattering state, w/ $E_b$, but it is surrounded by a continuum of scattering state energies.

If we want the probability of transition $|a\rangle$ to states in the vicinity of $|b\rangle$, we need to compute:

states near $|b\rangle$

$$P = \int_{E_b-\Delta E}^{E_b+\Delta E} \frac{|V_{a\bar{b}}|^2}{\hbar^2} \frac{\sin^2((\bar{\omega}_0-\omega)t/2)}{(\bar{\omega}_0-\omega)^2}\, \rho(\bar{E})\, d\bar{E}$$

$\bar{\omega}_0 = \frac{\bar{E}-E_a}{\hbar}$ for scatt. state energy $\bar{E}$

w/ $V_{a\bar{b}} = \langle a | V | \bar{b}\rangle$ scattering state

Here $\rho(\bar{E})$ is the "density of states" – the # of states w/ energy in the vicinity of $\bar{E}$ ($d\bar{n} = \rho(\bar{E})\,d\bar{E}$).

As in the discussion of interaction w/ incoherent light, we note that $\frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}$ is sharply peaked about $\omega_0$.

w/ $\omega_0 \equiv \frac{E_b-E_a}{\hbar}$, & so $\frac{|V_{a\bar{b}}|^2 \sin^2((\bar{\omega}_0-\omega)t/2)\rho(\bar{E})}{(\bar{\omega}_0-\omega)^2}$

will be large in the vicinity of $\omega_0$, so we can fix $|V_{ab}|^2\rho(E_b)$ & take those out of the integral, extending the limits to $\infty$:

$$P \approx \frac{|V_{ab}|^2}{\hbar^2}\rho(\omega_0)\int_0^\infty \frac{\sin^2((\bar{\omega}_0-\omega)t/2)}{(\bar{\omega}_0-\omega)^2}\, d\bar{E}$$

$d\bar{E} = \hbar\, d\bar{\omega}_0$

& we already saw this integral,

$$\int_0^\infty \frac{\sin^2((\bar{\omega}_0-\omega)t/2)}{(\bar{\omega}_0-\omega)^2}\, d\bar{\omega}_0 = \frac{\pi t}{2}$$

so $P = \frac{|V_{ab}|^2 \pi t}{2\hbar}\rho(E_b)$ w/ transition rate $R \equiv \frac{dP}{dt} = \frac{|V_{ab}|^2\pi}{2\hbar}\rho(E_b)$

## Density of States (Example)

In a typical scattering experiment, plane waves go in & plane waves come out.

What is the "density of states" for one-dimensional plane waves?

$\psi = \psi_0 e^{ikx}$ Focus on the imaginary piece: $\psi = \psi_0 \sin(kx)$

Imagine $\psi$ in a box of length $\ell$, where we know how to count energies, and use periodic b.c.

$$\psi(x+\ell) \doteq \psi_0 \sin(k(x+\ell)) = \psi_0 \sin(kx)$$

if $k\ell = 2\pi \cdot n$.

we have $k = \frac{2\pi}{\ell}\cdot n$, $\Rightarrow$ "$dn = \frac{\ell}{2\pi}dk$" (# of states in $dk$).

For a free particle, $k^2 = \frac{2mE}{\hbar^2}$, $\Rightarrow 2k\,dk = \frac{2m}{\hbar^2}dE$ so

$$dE = \frac{k\hbar^2}{m}\,dk.$$

The density of states is $\rho(E)$ w:

$$dn = \rho(E)\,dE \quad \text{, so here:}$$

$$\frac{\ell}{2\pi}\,dk = \rho(E)\,\frac{\hbar k^2}{m}\,dk$$

giving:

$$\rho(E) = \frac{m\ell}{2\pi\hbar^2 k} = \frac{m\ell}{2\pi\hbar\sqrt{2mE}}.$$

This expression depends on our choice of $\ell$ - in "many" problems the result will not depend on $\ell$, or we can usefully take $\ell \longrightarrow \infty$.

## Klein-Gordon Equation

Go back to the relativistic Hamiltonian:

$$H = \frac{mc^2}{\sqrt{1-v^2/c^2}} + U \quad w/ \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$$

so $H = mc^2\sqrt{1+\left(\frac{p}{mc}\right)^2} + U$

Issue: $p \longrightarrow \frac{\hbar}{i}\nabla$ is hard to evaluate under the square root.

Instead, make the operator by squaring:

$$(E-U)^2\psi = m^2c^4\left(1 - \frac{\hbar^2}{m^2c^2}\nabla^2\right)\psi$$

In $D=1$, this becomes:

$$-\hbar^2c^2\psi'' = (E-U(x))^2\psi - m^2c^4\psi.$$