Incoherent Perturbation

For the incident plane wave:

$$\vec{E} = E_0 \cos(\omega t)\hat{z} \qquad \vec{B} = \frac{E_0}{c}\cos(\omega t)\hat{x} \qquad (*)$$

the energy density is: $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$

$$= \epsilon_0 E_0^2 \cos^2(\omega t)$$

the time average is: $\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2 \equiv u$ ← to avoid notational clutter

Going back to our transition probability:

$$P(t) = \frac{E_0^2}{\hbar^2}|p'|^2 \frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}$$

↑ dipole matrix elt.

we can rewrite in terms of u:

$$P(t) = \frac{2u|p'|^2}{\epsilon_0\hbar^2}\frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}$$

Our model in $(*)$, is monochromatic light polarized in the $\hat{z}$ direction we'd like to model a source w/ multiple freq.s & polarizations.

1. Suppose the source has freq.-dep. energy density

$$u = \underbrace{\rho(\omega)d\omega}_{\text{energy density of source in vicinity of } \omega}$$

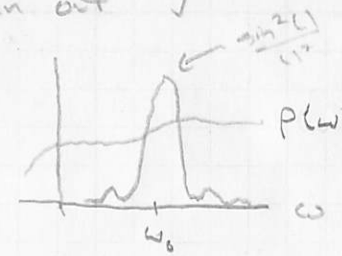
we'll sum over all freq.s

$$P(t) = \frac{2|p'|^2}{\epsilon_0\hbar^2}\int_0^\infty \rho(\omega)\underbrace{\frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}}_{\text{sharply peaked about } \omega_0} d\omega$$

focusing on the peak in the integrand near ω_0, we can replace

$$\rho(\omega) \to \rho(\omega_0)$$

inside the integral, at which point it can be taken out



$$P(t) = \frac{2|p'|^2}{\epsilon_0\hbar^2}\underbrace{\int_0^\infty \frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}d\omega}_{=\pi t/2}$$

$$= \frac{|p'|^2\rho(\omega_0)\pi}{\epsilon_0\hbar^2} t \quad \text{growing linearly in time.}$$

the rate, $R \equiv \frac{dP}{dt} = \frac{|p'|^2\rho(\omega_0)\pi}{\epsilon_0\hbar^2}$ of transition is constant.

Finally, we'll send $\langle\psi_b|qz|\psi_a\rangle \to \langle\psi_b|q\vec{r}|\psi_a\rangle$ & average over polarizations to get

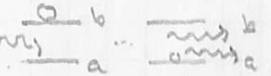
$$R = \frac{|\vec{p}'|^2\rho(\omega_0)\pi}{3\epsilon_0\hbar^2} \quad \text{w/ } \omega_0 \equiv \frac{E_b-E_a}{\hbar}$$

↳ avg. over polarizations

Einstein's A + B

For N particles in our 2-level system, focus on $N_b(t)$ – the number of particles in the b state at time t.

There are 3 processes to keep track of:

1. stimulated emission: 

the rate is $\sim \rho(\omega_0)$ as we just established + results in a loss.

$$dN_b = -\underbrace{[B_{ba}\rho(\omega_0)]}_{\text{transition rate for each particle}} \cdot \overset{\#\text{ of particles}}{N_b}\, \overset{\text{unit of time}}{dt}$$

so $\frac{dN_b}{dt} = -\left(B_{ba}\rho(\omega_0)\right)N_b$ w/ $B_{ba} = \frac{|\vec{\wp}|^2 \pi}{3\epsilon_0 \hbar^2}$

2. absorption 

here we gain particles from a – same rate!

$$dN_b = [B_{ab}\rho(\omega_0)]\, N_a\, dt$$

N_a: # of particles in a

(we know that $B_{ab} = B_{ba}$)

3. spontaneous emission – this one is new, + the argument we are going through in this accounting establishes its existence

for spontaneous emission: $dN_b = -A N_b dt$

A: rate of spontaneous emission

we are trying to find A.

Putting the 3 "mechanisms" together:

$$\frac{dN_b}{dt} = -B_{ba}\rho(\omega_0)N_b - AN_b + B_{ab}\rho(\omega_0)N_a$$

$$= \rho(\omega_0)\left[B_{ab}N_a - B_{ba}N_b\right] - AN_b$$

assume the system is in equilibrium, then

$$\frac{dN_b}{dt} = 0 \Rightarrow \rho(\omega_0) = \frac{A}{(N_a/N_b)B_{ab} - B_{ba}}$$

+ for equilibrium, $\frac{N_a}{N_b} = \frac{e^{-E_a/kT}}{e^{-E_b/kT}} = e^{\hbar\omega_0/kT}$ (at temp. T)

so

$$\rho(\omega_0) = \frac{A}{e^{\hbar\omega_0/kT}B_{ab} - B_{ba}} \quad (*)$$

the distribution here is Planck's black body radiation

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3 \left(e^{\hbar\omega_0/kT} - 1\right)}$$

comparing w/ (*), we "learn" that $B_{ab} = B_{ba}$ (already knew that one) +

$$A = \frac{\hbar\omega_0^3}{\pi^2 c^3} B_{ba}$$

We have $B_{ba} = B_{ab} = \frac{|\vec{p}|^2 \pi}{3 \epsilon_0 \hbar^2}$, &

$$A = \frac{|\vec{p}|^2 \omega_0^3}{3 \pi \epsilon_0 \hbar c^3}.$$

=

Lifetimes for Spontaneous Emission

If we just have spontaneous emission, (i.e. turn off all other lights)

$$\frac{dN_b}{dt} = -A N_b \Rightarrow N_b(t) = N_0 e^{-At}$$

timescales for exponentials are defined by τ ∋

$$N_b(\tau) = e^{-1} N_0$$

so $\tau = 1/A$, the "lifetime"

If we have more than 2 levels, so that there are multiple decay rates, each w/ their own rate:

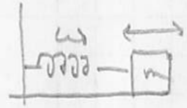
$$\frac{dN_b}{dt} = -A_1 N_b - A_2 N_b \cdots - A_n N_b$$

w/ $N_b(t) = N_0 e^{-[\sum_{j=1}^{n} A_j] t}$

w/ lifetime $\tau = [\sum_{j=1}^{n} A_j]^{-1}$

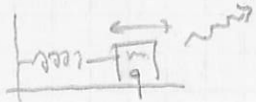
Harmonic Oscillator

Classical harmonic oscillator



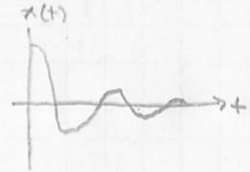
$$x \sim x_0 \cos(\omega t)$$

If the mass also carries charge,



$$m\ddot{x} = -m\omega_0^2 x - m\tau \dddot{x}$$

lose energy to radiation.



power lost is

$$P = \frac{q^2 x_0^2 \omega^4}{12 \pi \epsilon_0 c^3} \quad \text{(Larmor)}$$

& w/ $E = \frac{1}{2} m \omega^2 x_0^2$, $P = \frac{q^2 \omega^2 E}{6 \pi \epsilon_0 m c^3}$.