Oscillatory Perturbation

We had a perturbation $\hat{H}'(t) = V(\vec{r})\cos(\omega t)$ acting on a 2-level system w/ states $|\psi_a\rangle$ & $|\psi_b\rangle$, energies E_a & E_b.

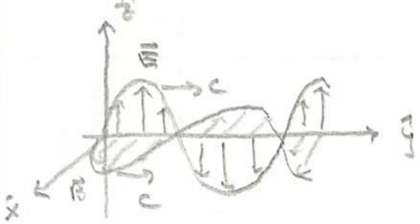
Starting from the a-state ($c_a(0)=1$, $c_b(0)=0$) we found the probability of transition to b is:

$$P(t) = \frac{|V_{ba}|^2}{\hbar^2} \frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2} \quad (\omega \approx \omega_0)$$

w/ $\omega_0 \equiv \frac{E_b - E_a}{\hbar}$, $V_{ba} \equiv \langle\psi_b|V(\vec{r})|\psi_a\rangle$

What causes an oscillatory, spatially varying perturbation?

E & M plane waves ("light"):



$$\vec{E} = E_0\cos(k(y-ct))\hat{z}$$

$$\vec{B} = \frac{E_0}{c}\cos(k(y-ct))\hat{x}$$

For a charge q, initially at rest at the origin

$$\vec{F} = qE_0\cos(\underbrace{kc}_{\omega}t)\hat{z}$$

causing the charge to oscillate at freq. ω

The associated potential is $V(z,t) = -E_0 z\cos(\omega t)$

so $\hat{H}' = -qE_0 z\cos(\omega t)$.

Now $V_{ba} = -E_0\langle\psi_b|\underbrace{qz}_{\text{dipole operator}}|\psi_a\rangle$

let $p' \equiv \langle\psi_b|qz|\psi_a\rangle$, then

$$P(t) = \frac{|p'|^2 E_0^2}{\hbar^2}\frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}.$$

example: for hydrogenic $|nlm\rangle$ states,

$\langle n'l'm'|\underbrace{q\vec{r}}_{\text{dipole op.}}|nlm\rangle = 0$ if $l+l'$ is even

so $\langle nlm|q\vec{r}|nlm\rangle = 0$ (the "$H'_{aa} = 0 = H'_{bb}$" assumption).

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Energy Transitions

The story we have in mind, classically, is

incoming radiation makes the charge accelerate causing it to radiate.

So far, we have a description of "absorption" in QM

$\rightsquigarrow$ $\underline{\circ e^-}_{\text{a-state}}$ leads to $\overline{\circ e^-}_{\text{b-state}}$ $\}$ $E_b - E_a = \hbar\omega_0$

$E_\gamma = \hbar\omega$

But the story works in the other direction, too — start w/ $c_a(0) = 0$, $c_b(0) = 1$

All that changes is $\hat{V}_{ba} \to V_{ab} = V_{ba}^*$ but the prob. only depends on $|V_{ba}|^2$ which is the same as $|V_{ab}|^2$.

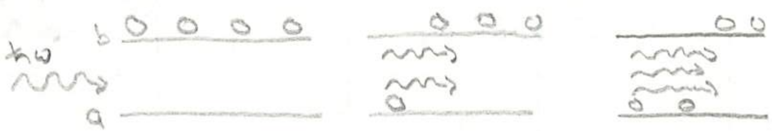
So you could also have:

$\hbar\omega \rightsquigarrow$ e^- b state leads to e^- a-state $\rightsquigarrow \hbar\omega_0$

"stimulated emission"

Note that here, you send in light w/ freq. $\omega \approx \omega_0$, but get out light at freq. $\omega_0 \equiv \frac{E_b - E_a}{\hbar}$

If you can prepare a bunch of charge in the b-state



a cartoon description of light amplification by the stimulated emission of radiation.

Incoherent Perturbation

Back to our plane wave w/ $\vec{E} = E_0 \cos(\omega t)\,\hat{z}$ & $\vec{B} = E_0/c \cos(\omega t)\,\hat{x}$.

the energy density is: $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 = \epsilon_0 E_0^2 \cos^2(\omega t)$

w/ time average: $\langle u \rangle \equiv \frac{1}{\underbrace{T}_{\frac{2\pi}{\omega}}} \int_0^T \epsilon_0 E_0^2 \cos^2(\omega t)\,dt = \frac{1}{2}\epsilon_0 E_0^2$

so our transition probability is:

$$P(t) = \frac{2u}{\epsilon_0 \hbar^2} |p'|^2 \frac{\sin^2((\omega_0 - \omega)t/2)}{(\omega_0 - \omega)^2}$$

(either $a \to b$ or $b \to a$)

Suppose we know the energy density of the light as a func. of its freq.:

$$u = \rho(\omega)\,d\omega$$

Then we can integrate the transition probability over all frequencies:

$$P(t) = \frac{2|p'|^2}{\epsilon_0 \hbar^2} \int_0^\infty \rho(\omega) \frac{\sin^2((\omega_0 - \omega)t/2)}{(\omega_0 - \omega)^2}\,d\omega$$

contributes primarily at $\omega \approx \omega_0$ due to

$$\approx \frac{2|p'|^2}{\epsilon_0 \hbar^2} \rho(\omega_0) \underbrace{\int_0^\infty \frac{\sin^2((\omega_0 - \omega)t/2)}{(\omega_0 - \omega)^2}\,d\omega}_{= \frac{\pi}{2} t}$$

$$P(t) = \frac{\pi |p'|^2}{\epsilon_0 \hbar^2} \rho(\omega_0) t$$

the rate of transition R, then, constant.

$$R \equiv \frac{dP}{dt} = \frac{\pi |p'|^2}{\epsilon_0 \hbar^2} \rho(\omega_0)$$

Finally - we have now a set of monochromatic plane waves of different frequencies - but we assumed they were all polarized in the same direction. If we average over polarizations:

$$R = \frac{\pi |\vec{p}'|^2}{3\epsilon_0 \hbar^2} \rho(\omega_0)$$